Numerical Analysis Qualifying Exam
Mathematics Department, New Mexico Tech
Spring 2009

(Answer all six problems)

1. Describe the solution of the least squares problem

$$\min_{\bar{x}\in R^n} \left\| A\bar{x} - \bar{b}_2 \right\|$$

by the QR factorization method where $A \in R^{m\times n}$, $m > n$ and the rank of A is known.

2. Develop the steepest descent method for solving the linear system, $A\bar{x} = \bar{b}$, with $A \in R^{n\times n}$, a positive definite matrix. Describe the selection of the search direction, the steplength, and discuss the update of the residual.

3. Let

$$f(x) = \frac{1 - \sin x}{\pi/2 - x}.$$

   (a) Explain why straight forward evaluation of $f(x)$ in double precision floating point arithmetic produces wildly inaccurate answers for x near $\pi/2$.

   (b) Derive an alternative formula that is much more accurate for x near $\pi/2$.

4. Consider the equation

$$x + \log x = 0.$$

Here log is the logarithm to the base e. This equation has a solution somewhere near $x = 0.5$. Derive a fixed point iteration scheme (other than the Newton's method) for solving this equation. Show that your fixed point iteration will converge if x_0 is sufficiently close to the root. Starting with $x_0 = 0.5$, use your iteration to solve the equation, obtaining a root accurate to 4 digits.

5. Given the ODE

$$y'(t) = f(t, y(t))$$
$$y(0) = y_0$$

   (a) Derive the Adams-Bashforth Two-Step Exlicit Method

$$w_{i+1} = w_i + \frac{h}{2}\left[3f(t_i, w_i) - f\left(t_{i-1,w_{i-1}}\right)\right]$$

   where $h > 0$, and $t_i = ih$, $i = 0, 1, \ldots$.

(b) Show that the truncation error is given by

$$\tau_{i+1} = \frac{5}{12} y''' (\mu_i) h^2$$

where $\mu_i \in (t_i, t_{i+1})$.

6. Let $f(x) \in C^4[a,b]$. For any $y, z \in R$, Simpson's rule is given by

$$S(y,z) = \frac{h}{3}\left[f(y) + 4f\left(\frac{y+z}{2}\right) + f(z)\right]$$

where $h = \frac{z-y}{2}$, and, in particular, satisfies

$$\int_a^b f(x)\,dx = S(a,b) - \frac{h^5}{90} f^{(4)}(\mu)$$

for some $\mu \in [a,b]$. The composite Simpson's rule satisfies

$$\int_a^b f(x)\,dx = S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right) - \frac{b-a}{180}\left(\frac{h}{2}\right)^4 f^{(4)}(\tilde{\mu})$$

for some $\tilde{\mu} \in [a,b]$, and where $h = \frac{b-a}{2}$.

(a) By equating the above relations and assuming that $f^{(4)}(\mu) = f^{(4)}(\tilde{\mu})$ derive an approximatation for the error

$$E(a,b) = \left| \int_a^b f(x)\,dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right|$$

involving $S(a,b)$, $S\left(a, \frac{a+b}{2}\right)$, and $S\left(\frac{a+b}{2}, b\right)$.

(b) Describe how an adaptive composite Simpson's algorithm can be developed from this error estimate to obtain an approximation to $\int_a^b f(x)\,dx$ to a desired accuracy of $\varepsilon > 0$. You don't have to be specific about the algorithm, just describe how it would work in general.