NEW MEXICO INSTITUTE OF MINING AND TECHNOLOGY

NONSTEADY FLOW TOWARD WELLS WHICH PARTIALLY PENETRATE THICK ARTESIAN AQUIFERS

by

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ABSTRACT

The nonsteady drawdown distribution for wells partially penetrating an artesian aquifer of infinite areal extent and unlimited thickness is solved. The confined aquifer is of uniform hydraulic conductivity and uniform specific storage. Two flow systems are considered: (1) wells just tapping an artesian aquifer and (2) wells penetrating an artesian aquifer. The solutions are obtained by the method of superposition using the solution of a spherical cavity located at (o, z_1); this basic solution is:

$$s_c = \frac{Q}{4\pi K}\left[\frac{\operatorname{erfc}\left(\sqrt{\frac{r^2 + (z - z_1)^2}{4\alpha t}}\right)}{\sqrt{r^2 + (z - z_1)^2}} + \frac{\operatorname{erfc}\left(\sqrt{\frac{r^2 + (z + z_1)^2}{4\alpha t}}\right)}{\sqrt{r^2 + (z + z_1)^2}}\right]$$

The variation of drawdown with time, and distance, for aquifer of unlimited thickness is compared with known solution for aquifer of limited thickness by means of graphs.

The function $F(u, x) = \int_u^\infty \frac{e^{-\beta}}{\beta} \operatorname{erf}(x\sqrt{\beta})\, d\beta$, involved in the mathematical solutions is evaluated numerically (table 1). Graphical methods based on the derived solutions are outlined for determining the formation constants. These methods are: (1) The

type-curve method, (2) the inflection-point method, and (3) the two-point method. Applications of the type-curve method are illustrated by treating data from Luna and Quay Counties, New Mexico.

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INTRODUCTION

General

Frequently, producing wells do not penetrate completely the aquifer from which they are pumping. The hydraulics of such wells is therefore different from wells which fully penetrate the aquifer.

The problem of partial penetration has long been recognized, and approximate solutions for various field conditions have been advanced. Most writers (see list of references) have assumed that the ground-water flow toward the well is stabilized; in other words, that a steady-state condition is attained. Such a condition rarely obtains during periods of actual well operation.

When dealing with nonsteady-flow problems of partial penetration, the current practice is to make use of the nonsteady-flow formulas for complete penetration either with or without adjustments. These adjustments, if used, are based on the steady-state flow formulas for partial penetration, depending on the presumably known characteristics of the aquifer. For example, if it is desired to compute the drawdown values in areas fairly close to the well, and if the horizontal permeability of the aquifer is very high, the bottom of the partially penetrated aquifer often is assumed to coincide with the bottom of the pumping well; that is, the flow condition is assumed to be one of complete penetration, the depth of the well equaling the thickness of the aquifer. It has also been customary to use the formulas of complete penetration to describe the flow in cases of partial penetration when (1) the drawdown observations are made in observation holes that completely penetrate the aquifer, (2) the drawdown values are required or observed at distances which are far away from the pumping well (that is, at distances such that the flow in the aquifer is more or less purely radial), or (3) the average of the drawdowns observed at the top and at the bottom of the aquifer is used in determining the formation coefficients.

Approximating the flow problems by procedures such as those mentioned above may give fair results in the case of an aquifer of relatively small thickness, provided that the conditions assumed in each case obtain. In the case, however, of extensive aquifers (aquifers of very great thickness), such procedures are not dependable. Either they may give erroneous results because of the approximate nature of the assumptions involved, or they may not be practical to carry out, or both.

Formulas for nonsteady flow toward a steady well partially penetrating an infinite artesian aquifer of limited thickness, and for nonsteady flow toward a well partially penetrating a thick water-table aquifer, have been developed by Hantush (1957) and Boulton (1954) respectively. Because of the nature of the assumptions made by Boulton, his formula cannot be used in the case of confined (artesian) flow; that of Hantush, on the other hand, describes the flow in leaky and nonleaky aquifers of both limited and unlimited thicknesses. The Hantush partial penetration formula can also be used, under certain conditions, to describe the flow in free aquifers. Computations using this formula in its present forms, although easily performed for aquifers of relatively small thicknesses, become lengthy and tedious in the case of very thick aquifers unless the infinite integral appearing in the second form of the solution is tabulated.

Purpose

The purpose of the present work is threefold:

1. To obtain solutions, for different flow conditions, for the drawdown distribution around a steady well that partially penetrates an infinite elastic artesian aquifer of unlimited depth.

2. To determine the degree of penetration in an aquifer of limited thickness, below which the flow pattern can be computed by the formula obtained for aquifers of unlimited depth.

3. To describe methods by which the formation constants can be obtained by aquifer pumping tests, when the appropriate solutions are used.

Statement of Problem

The problem is to determine the drawdown distribution around wells that are draining a thick artesian aquifer of infinite areal extent. It is assumed that the hydraulic conductivity and the specific storage remain constant both in space and time. Initially, the drawdown distribution is zero throughout the aquifer.

Two flow systems are considered:

Wells just tapping an artesian aquifer. In this system the water is discharged through a semispherical cavity at the bottom of the well, the radius of which is equal to that of the well bore. In practice such a flow system may represent flow toward: (a) wells of zero penetration and of constant discharge (fig. 1) or (b) artesian springs or wells of zero penetration and of constant head (fig. 2).

Wells penetrating an artesian aquifer. In this system the water is discharged through a cylinder whose length is equal to the depth of penetration. In considering this type of flow, two cases may arise in practice, that of flow toward a steadily discharging well, and that of a flowing well (well of constant drawdown). Only the first case is treated in this thesis (fig. 3).

THEORY

Major Symbols and Definitions

r Radial distance from the axis of the pumping well to any point in the space. (L)

z Depth of any point below the surface of the impermeable layer. (L)

z_1 Location of a spherical cavity at $r = o$. (L)

$R_1 = \sqrt{r^2 + z^2}$, the distance from the origin of the coordinate system to any point in space. (L)

$R = \sqrt{r^2 + (z - z_1)^2}$, the distance from a point located at z_1 to any point in space. (L)

R_w Effective radius of the spherical cavity. (L)

r_w Effective radius of the well. (L)

t Time since pumping started. (T)

t_i Time at which the inflection point takes place. (T)

$\bar{f}(p)$	Laplace transform, or the image, of $f(t)$.
b	Depth of the pumping well below the surface of the impermeable layer. (L)
b'	Length of the perforations in observation wells. (L)
s	Drawdown at any point (r, z) in the aquifer at any time t since startup. (L)
s_c	Drawdown at any point (r, z, t) in the aquifer due to a spherical cavity at the point (o, z_1).
s_a	Average drawdown in an observation well of perforated length b'. (L)
s_m	Maximum drawdown at any point (r, z). (L)
s_{am}	Maximum average drawdown in an observation well of perforated length b'. (L)
s_w	Drawdown at the pumping well at any time. (L)
s_i	Drawdown at the inflection point. (L)
Q	Rate of discharge of the well during pumping. (L^3/T)

K Hydraulic conductivity of the aquifer. (L/T)

S_s Specific storage, defined as the amount of water which a unit volume of the aquifer releases from storage under a unit head decline. (L^{-1})

α Ratio $\frac{K}{S_s}$. (L^2/T)

u Relation $\frac{r^2 S_s}{4Kt}$.

erf(x) Error function $= \frac{2}{\sqrt{\pi}} \int_0^x e^{-\beta^2} d\beta$.

erfc(x) Complementary error function $= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\beta^2} d\beta$.

$= 1 - \text{erf}(x)$

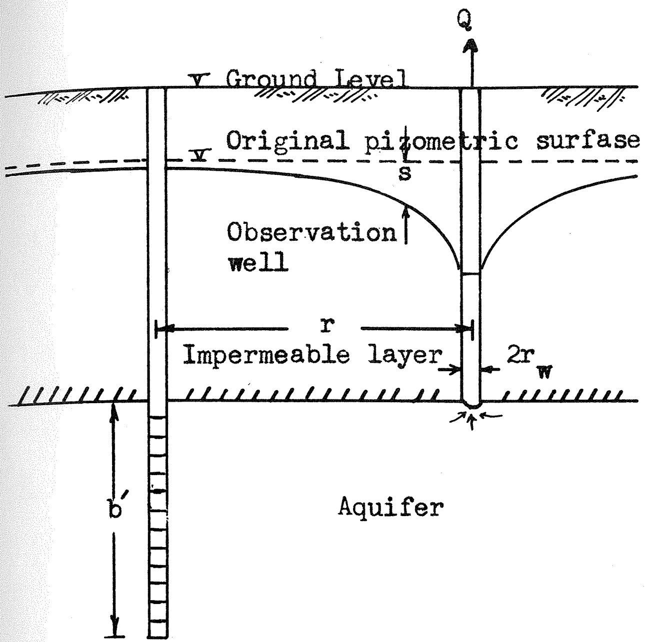


Fig-I Well just tapping the aquifer
case I -with constant discharge

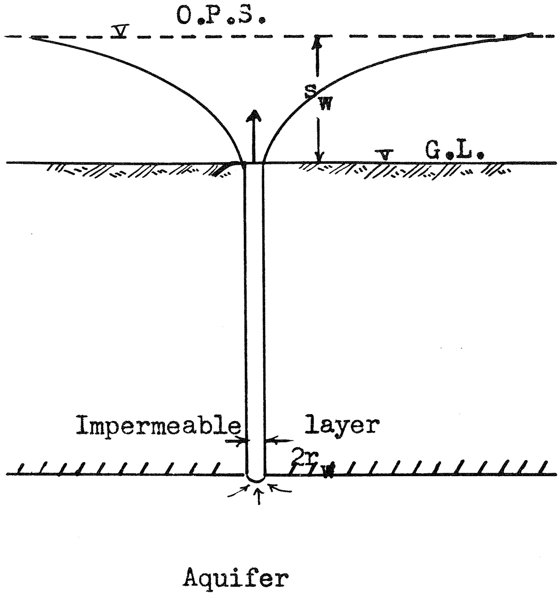


Fig-2 Well just tapping the aquifer
case 2-with constant drawdown

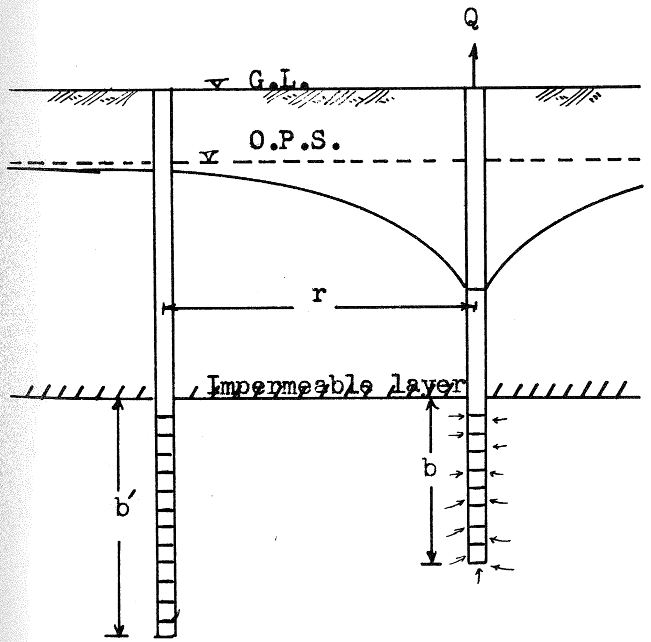


Fig-3 Well penetrating the aquifer

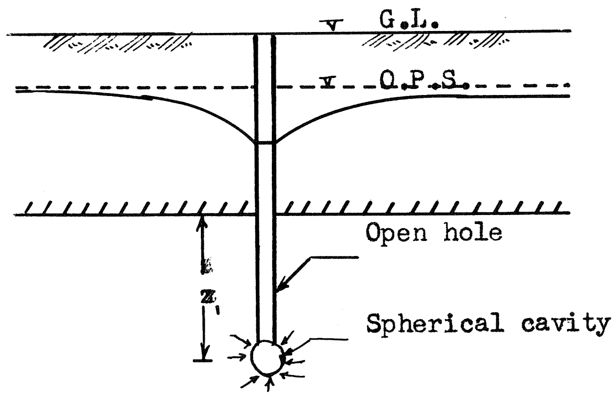


Fig-4 Flow toward a spherical cavity

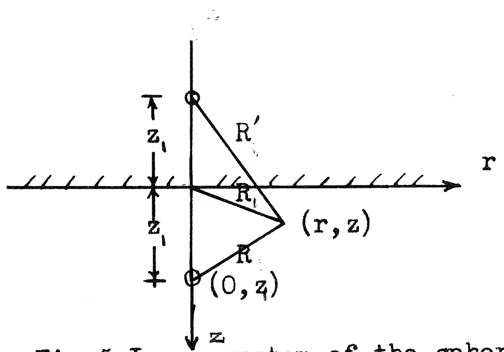


Fig-5 Image system of the spherical cavity

Differential Equation of Motion

The general differential equation of ground-water motion is

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \frac{\partial s}{\alpha \partial t}$$

In the cylindrical coordinates, the equation is

$$\frac{\partial^2 s}{\partial r^2} + \frac{\partial s}{r \partial r} + \frac{\partial^2 s}{r^2 \partial \theta^2} + \frac{\partial^2 s}{\partial z^2} = \frac{\partial s}{\alpha \partial t}$$

For purely spherical flow, the equation reduces to

$$\frac{\partial^2 s}{\partial R^2} + \frac{2 \partial s}{R \partial R} = \frac{\partial s}{\alpha \partial t}$$

For the derivation of these equations, the reader is referred to Jacob (1940), Muskat (1937), and others (see list of references).

Boundary-Value Problems and Solutions

Although several procedures can be employed in solving the problems treated below, the one followed here is believed to be the simplest and the most direct. The method of Laplace transformation is used to obtain a solution for the drawdown around a spherical cavity (fig. 4) draining an aquifer of infinite depth and of infinite areal extent. From this, the solutions of the different cases treated here are built up by the method of superposition.

Flow toward a spherical cavity

The system of flow being spherical, with the origin in this case taken at the center of the cavity (o, z_1), the boundary-value problem is:

$$\frac{\partial^2 s}{\partial R^2} + \frac{2\partial s}{R\partial R} = \frac{\partial s}{\alpha\partial t} \qquad (1)$$

$$s\ (R, o) = o \qquad (2)$$

$$s\ (\infty, t) = o \qquad (3)$$

$$\frac{\partial s}{\partial z}\ (r, o, t) = o \qquad (4)$$

where $R = \sqrt{r^2 + (z - z_1)^2}$, and $\alpha = \frac{K}{S_s}$.

By using the Laplace transformation with respect to (t) and applying boundary condition (2), the transformed boundary-value problem in (R, p) will be:

$$\frac{\partial^2 \bar{s}}{\partial R^2} + \frac{2\,\partial \bar{s}}{R\,\partial R} = \frac{p\bar{s}}{\alpha} \tag{5}$$

$$\bar{s}\,(\infty,\, p) \;=\; o \tag{6}$$

$$\frac{\partial \bar{s}}{\partial z}\,(r,\, o,\, p) = o \tag{7}$$

Equation (5) is reducible to the modified Bessel equation by the substitution of $\bar{s} = R^{-1/2}y$. After this substitution, equation (5) assumes the form

$$\frac{\partial^2 y}{\partial R^2} + \frac{\partial y}{R\,\partial R} - \left(\frac{1}{4R^2} + \frac{p}{\alpha}\right) y \;=\; o \tag{8}$$

Conditions (5), (6), and (7) are satisfied by

$$\bar{s}_c = c_1 \left[\frac{K_{1/2}\left(\sqrt{r^2 + (z - z_1)^2}\cdot\sqrt{\frac{p}{\alpha}}\right)}{\left[r^2 + (z - z_1)^2\right]^{1/4}} + \frac{K_{1/2}\left(\sqrt{r^2 + (z + z_1)^2}\cdot\sqrt{\frac{p}{\alpha}}\right)}{\left[r^2 + (z + z_1)^2\right]^{1/4}} \right] \tag{9}$$

where $K_{1/2}$ is the half-ordered modified Bessel function of the second kind.

By using the relation (Watson, 1944) $K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} \cdot e^{-z}$, equation (9) can be put in the form

$$\bar{s}_c = c \left[\frac{\exp\left[-\sqrt{\frac{p}{\alpha}\{r^2 + (z - z_1)^2\}}\right]}{\sqrt{r^2 + (z - z_1)^2}} + \frac{\exp\left[-\sqrt{\frac{p}{\alpha}\{r^2 + (z + z_1)^2\}}\right]}{\sqrt{r^2 + (z + z_1)^2}} \right] \qquad (10)$$

The proper distribution around the spherical cavity will depend on the value of the constant (c) appearing in equation (10), which in turn depends on the boundary conditions of the case to be considered.

Having obtained the elementary solution as given by equation (10), the solutions to the different cases follow.

Spherical Cavity

The boundary-value problem of this case is given by equations (1), (2), (3), (4), and the following condition:

$$4\pi K R^2 \frac{\partial s}{\partial R} = -Q \qquad (11)$$

$$R \longrightarrow 0$$

By applying the Laplace transformation on (11), the transformed boundary condition in (R, p) will be:

$$4\pi K R^2 \frac{\partial \bar{s}}{\partial R} = -\frac{Q}{p} \qquad (12)$$

$$R \longrightarrow o$$

The value of the constant (c) is obtained by applying condition (12) to equation (10), which gives $c = \frac{Q}{4\pi K p}$. Thus the solution can finally be written as

$$\bar{s} = \frac{Q}{4\pi K p}\left[\frac{\exp\cdot\left[-\sqrt{\frac{p}{\alpha}\{r^2 + (z - z_1)^2\}}\right]}{\sqrt{r^2 + (z - z_1)^2}} + \frac{\exp\left[-\sqrt{\frac{p}{\alpha}\{r^2 + (z + z_1)^2\}}\right]}{\sqrt{r^2 + (z + z_1)^2}}\right] \qquad)13)$$

(a) Nonsteady solution

By using tables of inverse Laplace transformation (Churchill, 1958), the drawdown distribution will be given by

$$s_c = \frac{Q}{4\pi K}\left[\frac{\operatorname{erfc}\sqrt{\frac{r^2 + (z - z_1)^2}{4\alpha t}}}{\sqrt{r^2 + (z - z_1)^2}} + \frac{\operatorname{erfc}\sqrt{\frac{r^2 + (z + z_1)^2}{4\alpha t}}}{\sqrt{r^2 + (z + z_1)^2}}\right] \qquad (14)$$

Average drawdown in wells of perforated depth: Equation (14) gives the drawdown at any point in the aquifer. The drawdown observed in wells of perforated section b' is the average value of the drawdowns at each point of the perforated length. Thus to obtain the average draw-down in a bore hole of length b', equation (14) is integrated with respect to z between the limits o and b', and divided by b'. The average drawdown s_{ca} can be obtained as follows:

$$s_{ca} = \frac{Q}{4\pi K b'}\left[\int_0^{b'} \left(\frac{\operatorname{erfc}\sqrt{\frac{r^2 + (z - z_1)^2}{4\alpha t}}}{\sqrt{r^2 + (z - z_1)^2}} + \frac{\operatorname{erfc}\sqrt{\frac{r^2 + (z + z_1)^2}{4\alpha t}}}{\sqrt{r^2 + (z + z_1)^2}} \right) dz \right]$$

$$= \frac{Q}{4\pi K b'}\left[\int_0^{b'} \frac{2}{\sqrt{\pi}} \left(\int_{\sqrt{\frac{r^2 + (z - z_1)^2}{4\alpha t}}}^{\infty} \frac{e^{-\beta^2}}{\sqrt{r^2 + (z - z_1)^2}}\, d\beta + \int_{\sqrt{\frac{r^2 + (z + z_1)^2}{4\alpha t}}}^{\infty} \frac{e^{-\beta_1^2}}{\sqrt{r^2 + (z + z_1)^2}}\, d\beta_1 \right) dz \right]$$

The substitution $\beta = \sqrt{r^2 + (z - z_1)^2}\, y$ and $\beta_1 = \sqrt{r^2 + (z + z_1)^2}\, y$

gives $d\beta = \sqrt{r^2 + (z - z_1)^2}\, dy$, and $d\beta_1 = \sqrt{r^2 + (z + z_1)^2}\, dy$

Therefore

$$s_{ca} = \frac{Q}{4\pi K b'}\left[\frac{2}{\sqrt{\pi}} \int_{\frac{1}{\sqrt{4\alpha t}}}^{\infty} e^{-r^2 y^2}\, dy \left(\int_0^{b'} e^{-(z - z_1)^2 y^2}\, dz + \int_0^{b'} e^{-(z + z_1)^2 y^2}\, dz \right) \right]$$

The substitution $(z - z_1)y = x$ and $(z + z_1)y = x_1$ gives $y\, dz = dx = dx_1$.

$$s_{ca} = \frac{Q}{4\pi K b'}\left[\frac{2}{\sqrt{\pi}} \int_{\frac{1}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-r^2y^2}}{y}\, dy \left(\int_{-z_1 y}^{(b'-z_1)y} e^{-x^2}\, dx + \int_{z_1 y}^{(b'+z_1)y} e^{-x_1^2}\, dx_1 \right)\right]$$

$$= \frac{Q}{4\pi K b'}\left[\int_{\frac{1}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-r^2y^2}}{y}\, dy \left[\ \mathrm{erf}\ (b'-z_1)y + \mathrm{erf}\ (b'+z_1)y\right]\right]$$

The substitution $r^2y^2 = \beta$ gives $2r^2ydy = d\beta$. The solution can finally be put in the form

$$s_{ca} = \frac{Q}{8\pi K b'}\left[\int_{u}^{\infty} \frac{e^{-\beta}}{\beta} \left[\mathrm{erf}\ \left(\frac{b'-z_1}{r}\sqrt{\beta}\right) + \mathrm{erf}\ \left(\frac{b'+z_1}{r}\sqrt{\beta}\right)\right] d\beta\right] \qquad (15)$$

(b) Steady-state solution

Strictly speaking, a steady-state solution is never attained, as an infinite aquifer is a closed reservoir into which flow from other sources does not take place. However, at any given finite distance from the center of the spherical cavity, however large it may be, the drawdown may, for all practical purposes, attain a constant value after a relatively large period of pumping (theoretically the drawdown is still changing, though at a very slow rate). Thus as time becomes greater (provided R remains finite, however large it may be), the steady state is approached. If t goes to infinity, equation (14) reduces to

$$s_{cm} = \frac{Q}{4\pi K}\left[\frac{1}{\sqrt{r^2 + (z - z_1)^2}} + \frac{1}{\sqrt{r^2 + (z + z_1)^2}}\right] \qquad (16)$$

The average steady drawdown in wells of perforated section can be obtained by integrating equation (16) with respect to z, from the limits o to b', and dividing by b':

$$s_{cam} = \frac{Q}{4\pi K b'}\left[\int_o^{b'} \left(\frac{1}{\sqrt{r^2 + (z - z_1)^2}} + \frac{1}{\sqrt{r^2 + (z + z_1)^2}}\right) dz\right]$$

$$= \frac{Q}{4\pi K b'}\left[\sinh^{-1}\left(\frac{b' - z_1}{r}\right) + \sinh^{-1}\left(\frac{b' + z_1}{r}\right)\right] \qquad (17)$$

Wells Just Tapping the Aquifer

Case 1. Wells of constant discharge

The boundary-value problem of this case is given by equations (1), (2), (3), (4), (11), and the following condition:

$$z_1 = 0 \qquad (18)$$

(a) Nonsteady solution

By applying condition (18) to equation (14), the solution can be found as

$$s = \frac{Q}{2\pi K} \cdot \frac{1}{R_1} \operatorname{erfc}\left(\frac{R_1}{\sqrt{4\alpha t}}\right) \qquad (19)$$

Average drawdown in wells of perforated depth: The average drawdown in a bore hole of length b', can be obtained by applying condition (18) to equation (15). Thus the solution can be put in the form

$$s_a = \frac{Q}{4\pi K b'} F(u, \frac{b'}{r}) \qquad (20)$$

where $F(u, \frac{b'}{r}) = \int_u^{\infty} \frac{e^{-\beta}}{\beta} \operatorname{erf}\left(\frac{b'}{r}\sqrt{\beta}\right) d\beta$

(b) Steady-state solution

As stated before, the steady-state can be approached provided that R_1 remains finite, however large it may be. Thus if t goes to infinity, equation (19) reduces to

$$s_m = \frac{Q}{2\pi K R_1} \tag{21}$$

The average steady drawdown in wells of perforated section b', can be obtained by applying condition (18) to equation (17). The solution can be found as

$$s_{am} = \frac{Q}{2\pi K b'} \sinh^{-1}\left(\frac{b'}{r}\right) \tag{22}$$

Case 2. Wells of constant drawdown

The boundary-value problem of this case is given by equations (1), (2), (3), (4), and the following two conditions:

$$s_1 = o \tag{23}$$

$$s\ (R_w, t) = s_w \tag{24}$$

By applying the Laplace transformation on (23) and (24), the transformed conditions in (R, p) will be:

$$\bar{s}_1 = 0 \tag{25}$$

$$\bar{s}\,(R_w,\, p) = \frac{s_w}{p} \tag{26}$$

Applying condition (25) and then condition (26) to equation (10), one obtains as the value of the constant c:

$$c = s_w R_w \frac{e^{R_w\sqrt{\frac{p}{\alpha}}}}{p}$$

Therefore, the transformed solution is:

$$\bar{s} = \frac{s_w R_w}{R_1} \; \frac{e^{-(R_1 - R_w)\sqrt{\frac{p}{\alpha}}}}{p} \tag{27}$$

where R_w is the effective radius of the spherical cavity at the bottom of the well.

(a) Nonsteady solution

By using tables of the inverse Laplace transform (Churchill 1958), the solution can finally be written as:

$$s = \frac{s_w \cdot R_w}{R_1} \cdot \operatorname{erfc} \frac{(R_1 - R_w)}{\sqrt{4\alpha t}} \tag{28}$$

Discharge of the well

The discharge of the well at any time t is given by:

$$Q = -2\pi K R_w^{2} \cdot \frac{ds}{dR_1}, \text{ at } (R_w, t) \tag{29}$$

By finding $\frac{ds}{dR_1}$ (R_w, t) from equation (28) and substituting in equation (29), the discharge variation will be given by:

$$Q = 2\pi K s_w R_w \left[\frac{2}{\sqrt{\pi}} \cdot \frac{R_w}{\sqrt{4\alpha t}} + 1 \right] \tag{30}$$

Average drawdown in wells of perforated depth:

By integrating equation (28) with respect to s between the limits o and b', dividing by b', the average drawdown s_a in a well of perforated section b', can be found as:

$$s_a = \frac{s_w \cdot R_w}{b'} \int_o^{b'} \frac{\operatorname{erfc}\left(\frac{R_1 - R_w}{\sqrt{4\alpha t}}\right)}{R_1} \, dz \tag{31}$$

(b) Steady-state solution

As stated in case 1, the steady state can be approached, provided that R_1 remains finite, however large it may be. Thus if t goes to infinity, equation (28) will reduce to:

$$s_m = \frac{s_w R_w}{R_1} \tag{32}$$

The average steady-state drawdown in an observation well of perforated length b' can be obtained by integrating equation (32) with respect to z between the limits o and b', and dividing by b':

$$s_{ma} = \frac{s_w R_w}{b'} \sinh^{-1} \frac{b'}{r} \tag{33}$$

The minimum discharge of the well is that of the steady state. From equation (30), it is:

$$Q_m = 2\pi K s_w R_w \tag{34}$$

Wells of Finite Depth

The solution for this case is that of spherical cavities each having a discharge $\frac{Q}{b}$ distributed continuously along the whole depth (b) of the well. Thus the boundary-value problem is that given by equations (1), (2), (3), (4), and the following conditions:

$$\lim_{R \to o} 4\pi K R^2 \frac{ds}{dR} = -\frac{Q}{b} \tag{35}$$

$$s = \int_o^b s_c \, dz_1 \tag{36}$$

where s_c is the drawdown due to a single spherical cavity located at point (o, z_1), as given by equation (10).

If the Laplace transformation is applied to (35) and (36), the transformed condition in (R, p) will be:

$$\lim_{R \to o} 4\pi K R^2 \frac{d\bar{s}}{dR} = -\frac{Q}{bp} \tag{37}$$

$$\bar{s} = \int_o^b \bar{s}_c \, dz_1 \tag{38}$$

By applying condition (37) to equation (10), the constant c can be found as:

$$c = \frac{Q}{4\pi Kbp}$$

With this value of c, the transformed solution is:

$$\bar{s}_c = \frac{Q}{4\pi Kb}\left[\frac{1}{p}\cdot\frac{e^{-\sqrt{\frac{p}{\alpha}\left\{r^2+(z-z_1)^2\right\}}}}{\sqrt{r^2+(z-z_1)^2}} + \frac{1}{p}\cdot\frac{e^{-\sqrt{\frac{p}{\alpha}\left\{r^2+(z+z_1)^2\right\}}}}{\sqrt{r^2+(z+z_1)^2}}\right] \tag{39}$$

Applying condition (38) to equation (39), one obtains:

$$\bar{s} = \frac{Q}{4\pi Kb}\left[\int_o^b \frac{1}{p}\cdot\frac{e^{-\sqrt{\frac{p}{\alpha}\left\{r^2+(z-z_1)^2\right\}}}}{\sqrt{r^2+(z-z_1)^2}}\,dz_1 + \int_o^b \frac{1}{p}\cdot\frac{e^{-\sqrt{\frac{p}{\alpha}\left\{r^2+(z+z_1)^2\right\}}}}{\sqrt{r^2+(z+z_1)^2}}\,dz_1\right] \tag{40}$$

(a) Nonsteady solution

By using tables of the inverse Laplace transform, the draw-down distribution will appear as:

$$s = \frac{Q}{4\pi Kb}\left[\int_0^b \frac{\operatorname{erfc}\left(\sqrt{\frac{r^2+(z-z_1)^2}{4\alpha t}}\right)}{\sqrt{r^2+(z-z_1)^2}}\,dz_1 + \int_0^b \frac{\operatorname{erfc}\left(\sqrt{\frac{r^2+(z+z_1)^2}{4\alpha t}}\right)}{\sqrt{r^2+(z+z_1)^2}}\,dz_1\right]$$

which, by further simplification, can be expressed as:

$$s = \frac{Q}{8\pi Kb}\, E\left(u, \frac{b}{r}, \frac{z}{r}\right) \tag{41}$$

where $E\left(u, \frac{b}{r}, \frac{z}{r}\right) = \left[\int_u^\infty \frac{e^{-\beta}}{\beta} \cdot \operatorname{erf}\left(\frac{b+z}{r}\sqrt{\beta}\right)\, d\beta + \int_u^\infty \frac{e^{-\beta}}{\beta} \cdot \operatorname{erf}\left(\frac{b-z}{r}\sqrt{\beta}\right)\, d\beta\right]$

Average drawdown in wells of perforated depth:

By integrating equation (41) with respect to z between o and b', dividing by b', and simplifying, the average drawdown in a well perforated throughout its depth will be found to be:

$$s_a = \frac{Q}{8\pi K b} \; E(u, \frac{b}{r}, \frac{b'}{r}) \qquad (42)$$

where

$$E(u, \frac{b}{r}, \frac{b'}{r}) = \Bigg[\int_u^{\infty} \frac{b+b'}{b'} \cdot \frac{e^{-\beta}}{\beta} \cdot \operatorname{erf}\left(\frac{b+b'}{r}\sqrt{\beta}\right) \, d\beta$$

$$- \int_u^{\infty} \frac{b-b'}{b'} \cdot \frac{e^{-\beta}}{\beta} \cdot \operatorname{erf}\left(\frac{b-b'}{r}\sqrt{\beta}\right) \, d\beta$$

$$+ \frac{2r}{b' \; \pi u} \, e^{-u} \left[\exp\left\{-\left(\frac{b+b'}{r}\right)^2 u\right\} - \exp\left\{-\left(\frac{b-b'}{r}\right)^2 u\right\} \right]$$

$$- \frac{2r}{b'} \sqrt{\left(\frac{b+b'}{r}\right)^2 + 1} \cdot \operatorname{erfc} \sqrt{\left\{\left(\frac{b+b'}{r}\right)^2 + 1\right\} u}$$

$$+ \frac{2r}{b'} \sqrt{\left(\frac{b-b'}{r}\right)^2 + 1} \cdot \operatorname{erfc} \sqrt{\left\{\left(\frac{b-b'}{r}\right)^2 + 1\right\} u} \Bigg]$$

(b) Steady-state solution

As stated previously, a steady state can be approached as time becomes very large, provided that r remains finite. Thus, as u goes to zero, equation (41) reduces to:

$$s_m = \frac{Q}{4\pi Kb}\left[\sinh^{-1}\left(\frac{b+b'}{r}\right) + \sinh^{-1}\left(\frac{b-b'}{r}\right)\right] \qquad (43)$$

Similarly, equation (42) will reduce to:

$$s_{am} = \frac{Q}{4\pi Kb}\left[\frac{b+b'}{b'} \sinh^{-1}\left(\frac{b+b'}{r}\right) - \frac{b-b'}{b'} \sinh^{-1}\left(\frac{b-b'}{r}\right) - \frac{1}{b'}\sqrt{(b+b')^2 + r^2} + \frac{1}{b'}\sqrt{(b-b')^2 + r^2}\right] \qquad (44)$$

DISCUSSION

The nonsteady drawdown distributions around a steady well that penetrates an artesian aquifer, infinite in areal extent, have been obtained by Hantush (1957). For an artesian aquifer of thickness m, the solution is given by either of the following:

$$s = \frac{Q}{4\pi K m}\left[W(u) + \frac{2m}{\pi b}\sum_{n=1}^{\infty}\frac{1}{n}\cos\frac{n\pi z}{m}\cdot\sin\frac{n\pi b}{m}\cdot W_n\left(u, \frac{n\pi r}{m}\right)\right] \qquad (45)$$

$$\text{or } s = \frac{Q}{8\pi K b}\left[\int_u^{\infty}\frac{e^{-\beta}}{\beta}d\beta \left(\operatorname{erf}\frac{b-z}{r}\sqrt{\beta} + \operatorname{erf}\frac{b+z}{r}\sqrt{\beta}\right) + \sum_{n=1}^{\infty}\int_u^{\infty}\frac{e^{-\beta}}{\beta}d\beta \left\{\operatorname{erf}\left(\frac{2nm+b-z}{r}\sqrt{\beta}\right) - \operatorname{erf}\left(\frac{2nm-b+z}{r}\sqrt{\beta}\right) + \operatorname{erf}\left(\frac{2nm+b+z}{r}\sqrt{\beta}\right) - \operatorname{erf}\left(\frac{2nm-b-z}{r}\sqrt{\beta}\right)\right\}\right] \qquad (46)$$

The flow pattern around a well partially penetrating an artesian aquifer of limited thickness will, during the initial period of flow, closely approximate that of an aquifer of unlimited depth.

The length of the period during which the two flow patterns approach each other depends on the depth of penetration, the distance from the pumped well, and the formation coefficients.

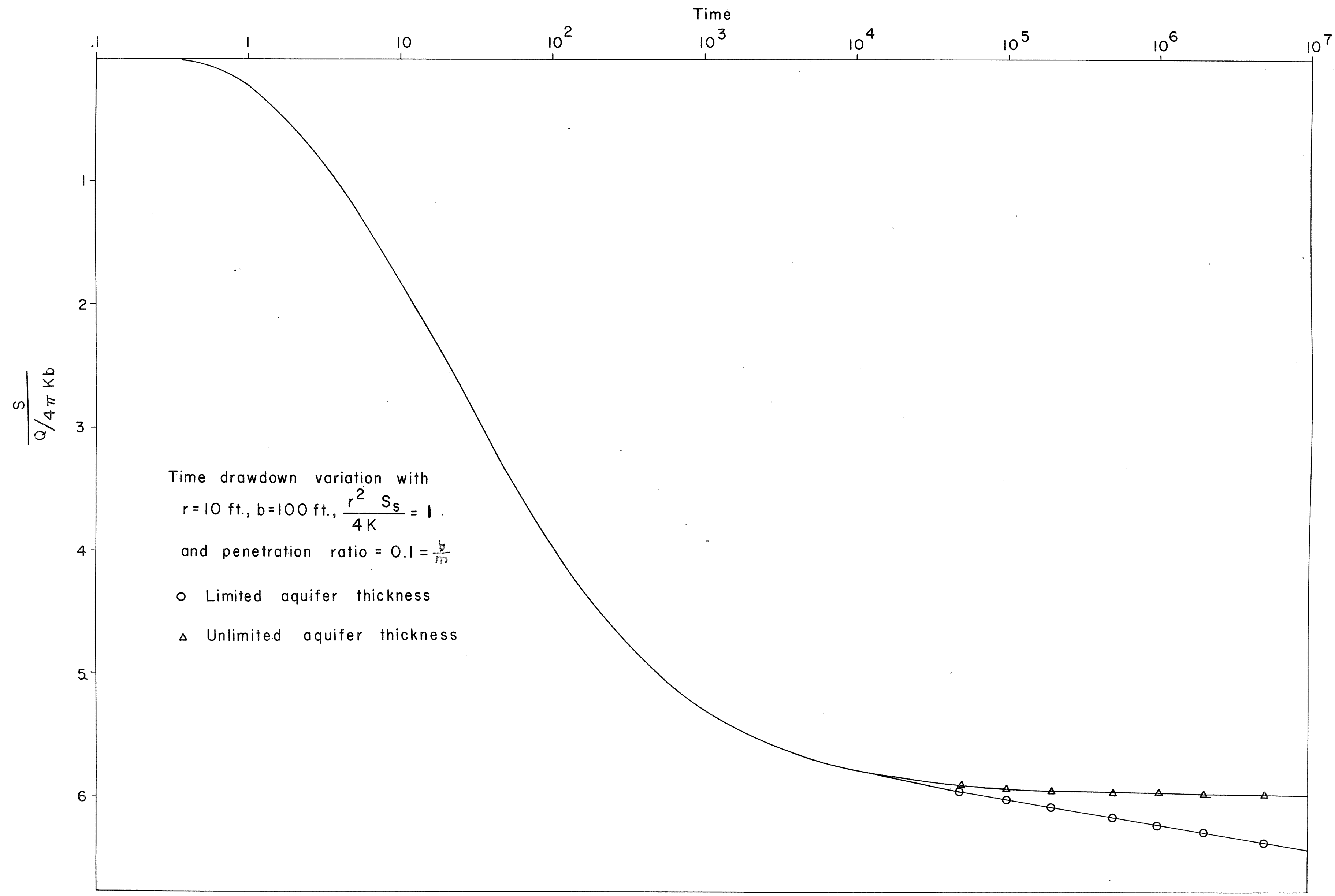
Computations in Hantush's solution and in equation (41) for $z = o$ show that for relatively short periods of pumping and for values of $\frac{b}{m} < .01$, the two solutions will yield the same results. Thus, for large values of m and/or small values of r (that is, for relatively small values of penetration and/or short distances from the pumped well), equation (41) can be used instead of equation (46), provided that the period of pumping is relatively short. Consequently, it is possible to use equation (41) for determining the formation coefficients by employing data collected during pumping tests on wells that partially penetrate an aquifer of unknown relatively large thickness. Equation (41) can also be used to predict declines of water level during short periods of well operation. For long term prediction, equations (45) or (46) should be used.

Comparing equation (42) with the Theis formula (1935) for an aquifer whose thickness is assumed to coincide with the bottom of the pumped well shows that the two solutions will approach each other during the early stages of pumping only if the distances are small. They deviate greatly from each other for large distances, although the general trend of the variation appears to be the same. The period during which the two solutions approach each other may range from a few minutes to several months, depending on the distance, thickness of the aquifer, and the formation coefficients. Thus, if it is known that the well partially penetrates a thick aquifer, analysis based on the Theis formula, the aquifer being assumed to end with the bottom of the well, may or may not give reliable results from data collected during a pumping test.

Figure 6 is constructed for a given set of the parameters involved, to illustrate the points discussed above. The values used are $r = 10$ ft, $b = 100$ ft, and $\frac{r^2 S_s}{4K} = 1$, and the ratio of penetration is 0.1.

Figure 7 is constructed to demonstrate the behavior of the drawdown curve given by equation (42) in comparison with the Theis formula for different values of r, the values used are $r = 1$, 10, 100 ft, and $u = \frac{8 \times 10^{-5}}{t} r^2$.

In Figure 8, the drawdown observed in a well that is perforated throughout its depth of saturation is compared with the drawdown that would have been observed had the same well been open at the top of the aquifer alone. It is assumed that $\frac{b}{r} = 10$ and that $\frac{r^2 S_s}{4K} = 1$. The same figure also shows the contrasts in drawdowns for different depths of the observed well. Thus, in analyzing data observed in a perforated well or in an open hole (except in the pumped well), equation (42), which gives the average drawdown, should be used instead of equation (41), which gives the drawdown at any point (r, z, t). Apart from the well losses, the water level in a pumped well that partially penetrates an aquifer will correspond to that experienced at the top of the aquifer. Therefore, equation (41), with $z = o$ and $r = r_w$, is to be used when predicting water levels in the pumped well or when analyzing data observed in such a well.



Time
.1
1
10
10^2
10^3
10^4
10^5
10^6
10^7
$\frac{s}{Q/4\pi Kb}$
1
2
3
4
5
6
Time drawdown variation with
r = 10 ft., b = 100 ft., $\frac{r^2 S_s}{4K} = 1$.
and penetration ratio = 0.1 = $\frac{b}{m}$
o Limited aquifer thickness
Δ Unlimited aquifer thickness

Figure 6

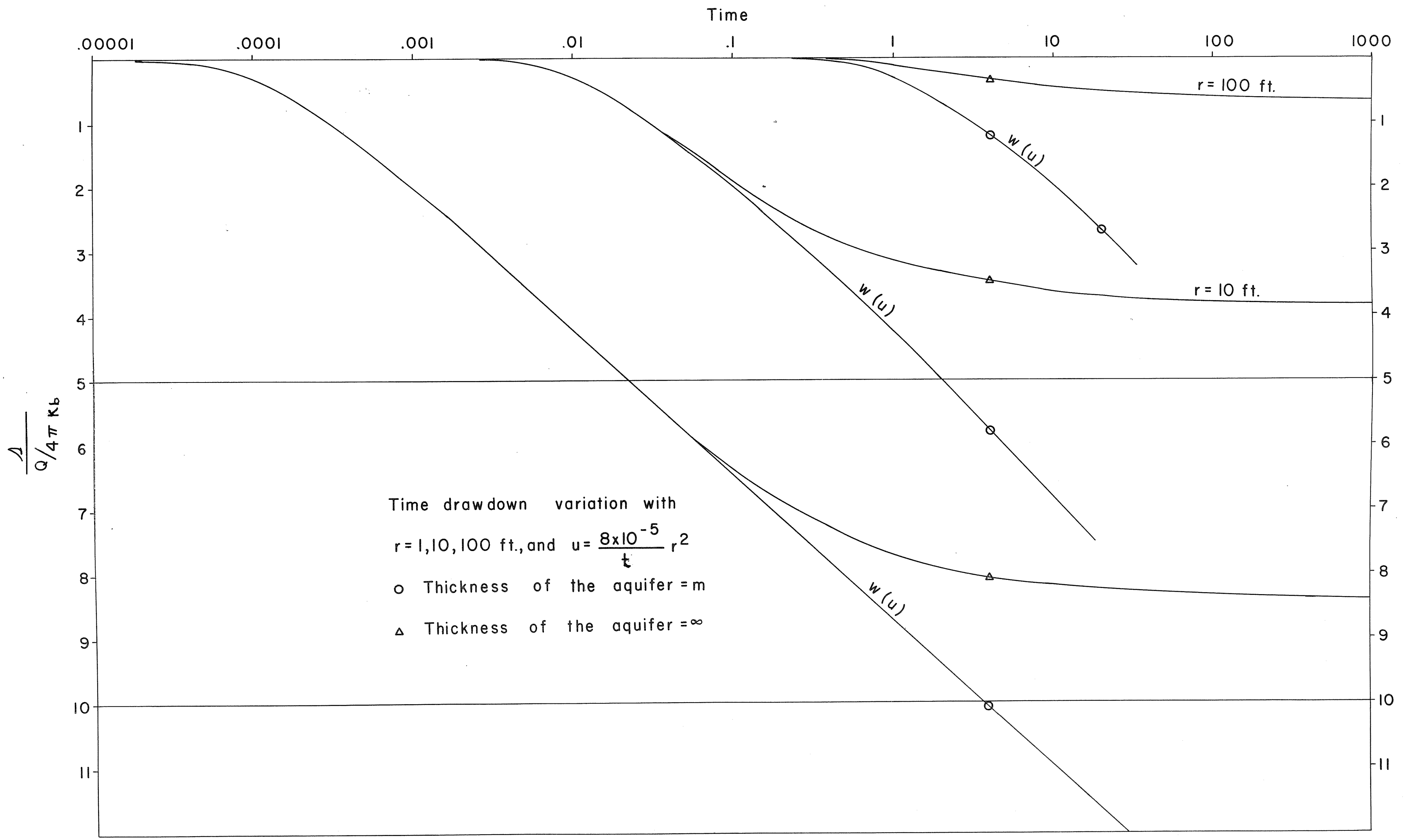

Time
.00001
.0001
.001
.01
.1
1
10
100
1000
1
2
3
4
5
6
7
8
9
10
11
r = 100 ft.
r = 10 ft.
w (u)
w (u)
w (u)
Time drawdown variation with
r = 1,10,100 ft., and u = 8x10^-5 / t r^2
o Thickness of the aquifer = m
Δ Thickness of the aquifer = ∞

Figure 7

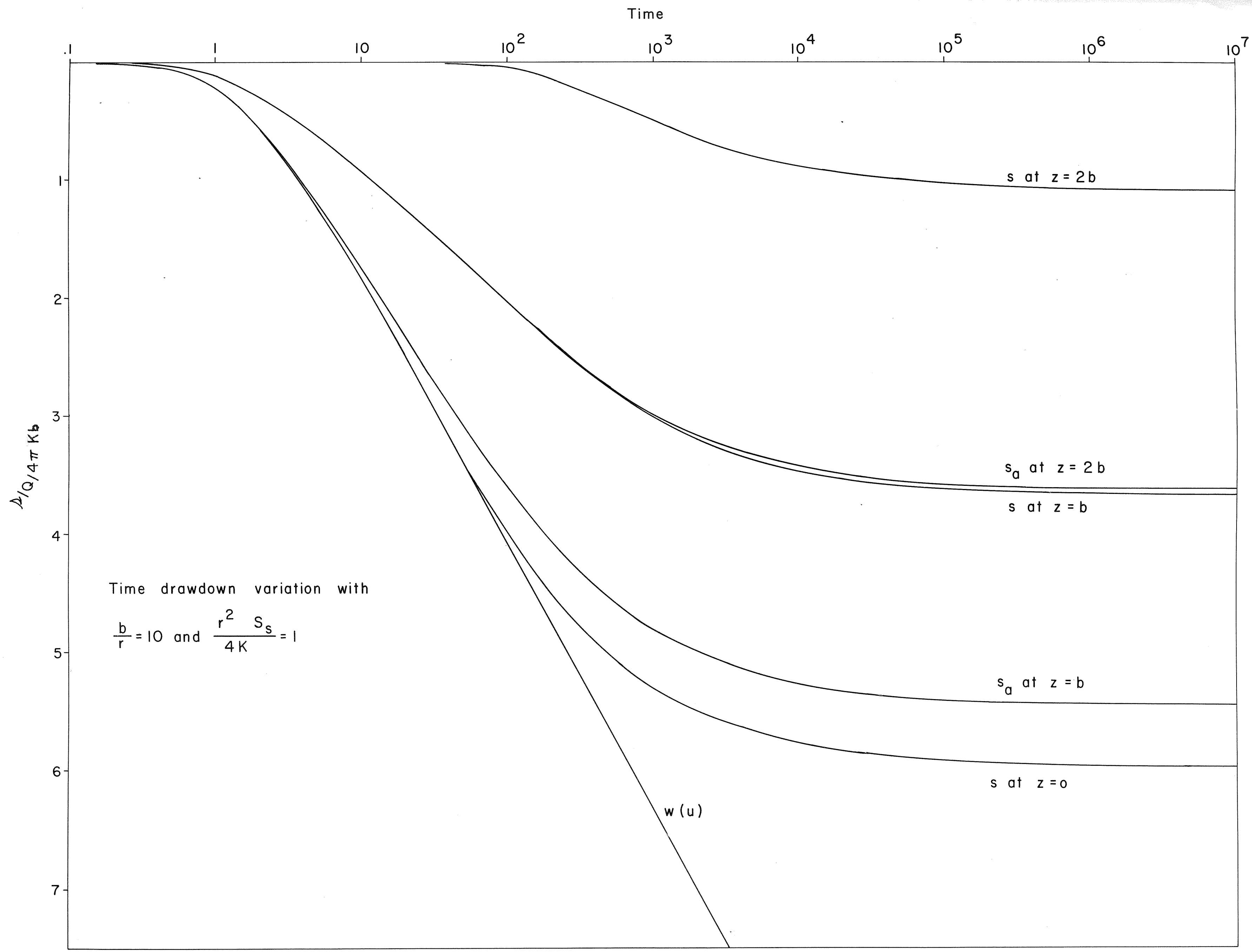

Time
.1
1
10
10²
10³
10⁴
10⁵
10⁶
10⁷
1
2
3
4
5
6
7
s/Q/4π Kb
s at z = 2b
s_a at z = 2b
s at z = b
s_a at z = b
s at z = o
w (u)
Time drawdown variation with
b/r = 10 and r² S_s / 4K = 1

Figure 8

TABLES OF FUNCTIONS

To make use of the solutions obtained in the present work, tabular values of the functions erfc (x), e^{-x}, $\sinh^{-1}(x)$, and

$$F(u, x) = \int_u^\infty \frac{e^{-\beta}}{\beta} \operatorname{erf}(x\sqrt{\beta})\, d\beta$$

should be available. The first three of these functions are available in the literature (Dwight, Mathematical Tables, Dover). For the function F (u, x), previously not available in tabular form, sufficient tabulation for all practical purposes is given in Table 1. It is obtained through numerical integration by using the trapezoidal rule. Although the table is not suitable for linear interpolation, it gives, however, a sufficient number of points to construct smooth curves (F versus u, and F versus x), that can be used to obtain intermediate values with sufficient accuracy. Figure 9 is a plot for F (u, x) versus x.

For $u \leq \frac{.01}{x^2} \geq 10$, the function F (u, x) can be approximated by

$$F(u, x) = (2x - \frac{x^3}{3})(\operatorname{erf}\frac{.1}{x} - \operatorname{erf}\sqrt{u}) - \frac{2}{\sqrt{\pi}}(\frac{x^3}{3}\sqrt{u}e^{-u} - \frac{x^2}{30}e^{-\frac{.01}{x^2}}) \qquad (47)$$

For $u \geq \frac{9}{x^2}$, the approximation is

$$F(u, x) = W(u) \tag{48}$$

where W (u) is the exponential integral, or what in the field of hydrology is known as the well function. The function has been tabulated in the literature (Wenzel, 1942; Wisler and Brater, 1940).

TABLE 1 — VALUES OF THE FUNCTION F (u, x)

$$F(u, x) = \int_u^\infty \frac{e^{-\beta}}{\beta} \operatorname{erf} x\sqrt{\beta} \; d\beta$$

u \ x	.001	.002	.003	.004	.005	.006	.007	.008	.009	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10
0	.002	.004	.006	.008	.010	.012	.014	.016	.018	.020	.040	.060	.080	.100	.120	.140	.160	.180	.200
10^{-8}	.002	.004	.006	.008	.010	.012	.014	.016	.018	.020	.040	.060	.080	.100	.120	.140	.160	.180	.200
5×10^{-8}	.002	.004	.006	.008	.010	.012	.014	.016	.018	.020	.040	.060	.080	.100	.120	.140	.160	.180	.200
10^{-7}	.002	.004	.006	.008	.010	.012	.014	.016	.018	.020	.040	.060	.080	.100	.120	.140	.160	.180	.200
5×10^{-7}	.002	.004	.006	.008	.010	.012	.014	.016	.018	.020	.040	.060	.080	.100	.120	.140	.160	.180	.200
10^{-6}	.002	.004	.006	.008	.010	.012	.014	.016	.018	.020	.040	.060	.080	.100	.120	.140	.160	.180	.200
5×10^{-6}	.002	.004	.006	.008	.010	.012	.014	.016	.018	.020	.040	.060	.080	.100	.120	.140	.160	.179	.199
10^{-5}	.002	.004	.006	.008	.010	.012	.014	.016	.018	.020	.040	.060	.080	.100	.120	.140	.159	.179	.199
5×10^{-5}	.002	.004	.0059	.0079	.0099	.012	.014	.016	.018	.020	.040	.060	.079	.099	.1190	.139	.159	.178	.198
10^{-4}	.002	.004	.0059	.0079	.0099	.012	.014	.016	.018	.020	.040	.059	.079	.099	.119	.138	.158	.178	.197
5×10^{-4}	.002	.0039	.0058	.0078	.0097	.012	.014	.016	.018	.020	.039	.058	.078	.097	.1170	.136	.156	.175	.195
10^{-3}	.0019	.0039	.0058	.0077	.0096	.012	.014	.015	.017	.019	.039	.058	.077	.096	.116	.135	.154	.173	.193
5×10^{-3}	.0019	.0037	.0055	.0074	.0092	.011	.013	.015	.017	.018	.037	.055	.074	.092	.110	.129	.147	.166	.184
10^{-2}	.0018	.0035	.0053	.0071	.0089	.011	.012	.014	.016	.018	.035	.053	.071	.089	.106	.124	.142	.159	.177
5×10^{-2}	.0015	.0030	.0045	.0060	.0075	.0090	.011	.012	.013	.015	.030	.045	.060	.075	.090	.105	.120	.135	.150
10^{-1}	.0013	.0027	.0040	.0053	.0066	.0080	.0093	.011	.012	.013	.027	.040	.053	.065	.079	.092	.105	.118	.131
5×10^{-1}	.0006	.0013	.0019	.0025	.0032	.0038	.0044	.0051	.0057	.0063	.013	.019	.025	.041	.049	.057	.065	.0730	.081
1	.0003	.0006	.0009	.0013	.0016	.0019	.0022	.0025	.0028	.0031	.0063	.0094	.012	.016	.019	.022	.025	.028	.031
5	.0000031	.0000063	.0000094	.000013	.000016	.000019	.000022	.000025	.000028	.000031	.000063	.000094	.00012	.00016	.00019	.00023	.00026	.00028	.00031
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE 1 continued

u \ x	.10	.20	.30	.40	.50	.60	.70	.80	.90	1	2	3	4	5	6	7	8	9	10
0	.200	.397	.591	.780	.962	1.138	1.305	1.465	1.618	1.763	2.887	3.637	4.189	4.625	4.984	5.288	5.553	5.787	5.996
10^{-8}	.200	.397	.591	.780	.962	1.138	1.305	1.465	1.617	1.762	2.885	3.636	4.189	4.624	4.983	5.287	5.551	5.785	5.994
5×10^{-8}	.200	.397	.591	.780	.962	1.138	1.305	1.465	1.617	1.762	2.885	3.635	4.188	4.622	4.981	5.285	5.549	5.783	5.991
10^{-7}	.200	.397	.591	.780	.962	1.137	1.305	1.465	1.617	1.762	2.884	3.634	4.187	4.621	4.980	5.284	5.547	5.781	5.989
5×10^{-7}	.200	.397	.591	.779	.962	1.137	1.304	1.464	1.616	1.761	2.882	3.632	4.183	4.617	4.974	5.277	5.540	5.773	5.980
10^{-6}	.200	.397	.591	.779	.962	1.137	1.304	1.464	1.616	1.760	2.881	3.630	4.180	4.614	4.970	5.273	5.533	5.767	5.974
5×10^{-6}	.199	.396	.590	.778	.960	1.135	1.302	1.461	1.613	1.758	2.876	3.622	4.169	4.600	4.954	5.253	5.512	5.742	5.946
10^{-5}	.199	.396	.589	.777	.959	1.134	1.300	1.460	1.611	1.756	2.871	3.616	4.161	4.589	4.941	5.239	5.496	5.723	5.925
5×10^{-5}	.198	.394	.587	.774	.954	1.128	1.294	1.453	1.603	1.747	2.854	3.583	4.126	4.545	4.888	5.177	5.425	5.644	5.837
10^{-4}	.197	.393	.585	.771	.951	1.124	1.290	1.447	1.597	1.740	2.841	3.569	4.099	4.512	4.849	5.131	5.372	5.584	5.771
5×10^{-4}	.195	.387	.576	.760	.937	1.108	1.270	1.425	1.572	1.712	2.785	3.486	3.988	4.373	4.682	4.937	5.151	5.335	5.495
10^{-3}	.193	.383	.570	.751	.927	1.095	1.255	1.408	1.554	1.691	2.743	3.423	3.905	4.269	4.558	4.792	4.986	5.151	5.291
5×10^{-3}	.184	.366	.544	.716	.883	1.042	1.194	1.338	1.474	1.603	2.568	3.161	3.558	3.839	4.046	4.203	4.321	4.414	4.484
10^{-2}	.177	.351	.520	.683	.841	.993	1.137	1.276	1.410	1.538	2.438	2.969	3.305	3.530	3.685	3.793	3.868	3.921	3.958
5×10^{-2}	.150	.297	.441	.581	.715	.841	.959	1.070	1.173	1.269	1.914	2.216	2.356	2.421	2.449	2.461	2.465	2.467	2.467
10^{-1}	.131	.259	.384	.504	.618	.725	.824	.917	1.002	1.080	1.560	1.739	1.799	1.817	1.822	1.823	1.823	1.823	1.823
5×10^{-1}	.081	.127	.188	.232	.288	.332	.371	.405	.434	.458	.551	.560	.560						.560
1	.031	.061	.090	.115	.137	.156	.171	.183	.193	.200	.219	.219							.219
5	.00031	.00058	.00079	.00095	.0010	.0011	.0011	.0011	.0011	.0011	.00115								.0011
10	0	0	0	0	0	0	0	0	.000012										.000012

TABLE I continued

u \ x	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
0	5.996	6.808	7.379	7.826	8.189	8.498	8.764	9.000	9.210	9.400	9.575	9.735	9.883	10.021	10.150	10.271	10.386	10.494	10.597
10^{-8}	5.994	6.801	7.375	7.820	8.182	8.489	8.755	8.990	9.199	9.389	9.562	9.721	9.868	10.004	10.133	10.253	10.366	10.473	10.574
5×10^{-8}	5.991	6.797	7.369	7.813	8.174	8.480	8.744	8.978	9.186	9.374	9.545	9.703	9.848	9.984	10.110	10.230	10.341	10.446	10.547
10^{-7}	5.989	6.794	7.365	7.808	8.168	8.472	8.736	8.968	9.175	9.362	9.532	9.689	9.834	9.968	10.093	10.212	10.322	10.423	10.525
5×10^{-7}	5.980	6.781	7.347	7.785	8.141	8.441	8.700	8.928	9.131	9.313	9.479	9.631	9.771	9.901	10.022	10.136	10.242	10.342	10.437
10^{-6}	5.974	6.771	7.334	7.769	8.121	8.418	8.674	8.898	9.098	9.277	9.440	9.564	9.726	9.852	9.970	10.081	10.183	10.280	10.371
5×10^{-6}	5.946	6.729	7.278	7.699	8.037	8.320	8.562	8.773	8.959	9.124	9.273	9.413	9.532	9.644	9.748	9.845	9.934	10.017	10.095
10^{-5}	5.925	6.698	7.237	7.647	7.975	8.248	8.479	8.679	8.855	9.010	9.149	9.274	9.387	9.490	9.584	9.671	9.739	9.823	9.891
5×10^{-5}	5.837	6.565	7.061	7.428	7.713	7.943	8.132	8.290	8.424	8.538	8.636	8.722	8.796	8.861	8.917	8.967	9.010	9.049	9.082
10^{-4}	5.771	6.467	6.928	7.265	7.519	7.718	7.877	8.006	8.112	8.200	8.272	8.333	8.383	8.425	8.460	8.491	8.515	8.535	8.552
5×10^{-4}	5.495	6.057	6.392	6.606	6.746	6.840	6.903	6.945	6.973	6.985	7.004	7.012	7.017	7.020	7.022	7.024	7.024	7.024	7.024
10^{-3}	5.921	5.760	6.001	6.153	6.233	6.278	6.304	6.318	6.325	6.328	6.330	6.331	6.332	6.332	6.332	6.332	6.332	6.332	6.332
5×10^{-3}	4.484	4.662	4.712	4.724	4.726	4.726	4.726	4.726											4.726
10^{-2}	3.958	4.028	4.037	4.038	4.038														4.038
5×10^{-2}	2.467	2.468																	2.468
10^{-1}	1.823																		1.823
5×10^{-1}	.560																		.560
1	.219																		.219
5	.0011																		.0011
10	.000012																		.000012

TABLE I continued

u \ x	100	150	200	250	300	350	400	450	500	550	600	650	700	750	800	850	900	950	1000
0	10.597	11.406	11.983	12.430	12.794	13.102	13.369	13.604	13.815	14.006	14.180	14.340	14.488	14.627	14.755	14.877	14.991	15.099	15.202
10^{-8}	10.574	11.379	11.938	12.373	12.726	13.023	13.279	13.503	13.703	13.882	14.045	14.194	14.331	14.457	14.585	14.686	14.789	14.885	14.976
5×10^{-8}	10.547	11.332	11.881	12.302	12.640	12.924	13.168	13.378	13.564	13.729	13.878	14.018	14.137	14.250	14.353	14.450	14.539	14.622	14.700
10^{-7}	10.525	11.300	11.840	12.251	12.580	12.853	13.099	13.284	13.460	13.615	13.754	13.879	13.992	14.094	14.189	14.276	14.344	14.428	14.496
5×10^{-7}	10.437	11.168	11.665	12.032	12.317	12.548	12.737	12.895	13.029	13.143	13.241	13.327	13.401	13.465	13.522	13.572	13.615	13.654	13.687
10^{-6}	10.371	11.070	11.534	11.870	12.123	12.323	12.482	12.611	12.717	12.805	12.877	12.928	12.979	13.030	13.065	13.095	13.120	13.140	13.157
5×10^{-6}	10.095	10.660	10.996	11.210	11.350	11.444	11.508	11.550	11.578	11.597	11.609	11.617	11.622	11.624	11.626	11.628	11.628	11.628	11.629
10^{-5}	9.891	10.363	10.615	10.757	10.830	10.883	10.908	10.922	10.929	10.932	10.934	10.935	10.936	10.936	10.936	10.936	10.936	10.936	10.936
5×10^{-5}	9.082	9.262	9.312	9.324	9.326	9.326	9.326	9.326											9.326
10^{-4}	8.552	8.623	8.632	8.633	8.633														8.633
5×10^{-4}	7.024	7.024																	7.024
10^{-3}	6.332																		6.332
5×10^{-3}	4.726																		4.726
10^{-2}	4.038																		4.038
5×10^{-2}	2.468																		2.468
10^{-1}	1.823																		1.823
5×10^{-1}	.560																		.560
1	.219																		.219
5	.0011																		.0011
10	.000012																		.000012

TABLE I continued

u \ x	1000	1500	2000	2500	3000	3500	4000	4500	5000	5500	6000	6500	7000	7500	8000	8500	9000	9500	10,000	15,000	20,000	25,000	30,000
0	15.202	16.013	16.588	17.035	17.399	17.707	17.975	18.210	18.421	18.611	18.785	18.945	19.094	19.232	19.361	19.482	19.596	19.705	19.807	20.618	21.194	21.640	22.005
10^{-8}	14.976	15.675	16.139	16.475	16.729	16.928	17.087	17.217	17.322	17.410	17.482	17.543	17.584	17.635	17.670	17.701	17.725	17.745	17.762	17.833	17.843	17.843	17.843
5×10^{-8}	14.700	15.265	15.601	15.815	15.955	16.049	16.113	16.155	16.183	16.202	16.214	16.222	16.227	16.230	16.231	16.233	16.233	16.233	16.234	16.234			16.234
10^{-7}	14.496	14.968	15.220	15.362	15.442	-5.487	15.513	15.527	15.534	15.538	15.540	15.540	15.541	15.541	15.541	15.541	15.541	15.541					15.541
5×10^{-7}	13.687	13.868	13.917	13.929	13.931	13.931	13.931	13.931															13.931
10^{-6}	13.157	13.228	13.238	13.328	13.328																		13.238
5×10^{-6}	11.629	11.629																					11.629
10^{-5}	10.936																						10.936
5×10^{-5}	9.326																						9.326
10^{-4}	8.633																						8.633
5×10^{-4}	7.024																						7.024
10^{-3}	6.332																						6.332
5×10^{-3}	4.726																						4.726
10^{-2}	4.038																						4.038
5×10^{-2}	2.468																						2.468
10^{-1}	1.823																						1.823
5×10^{-1}	.560																						.560
1	.219																						.219
5	.0011																						.00115
10	.000012																						.000012

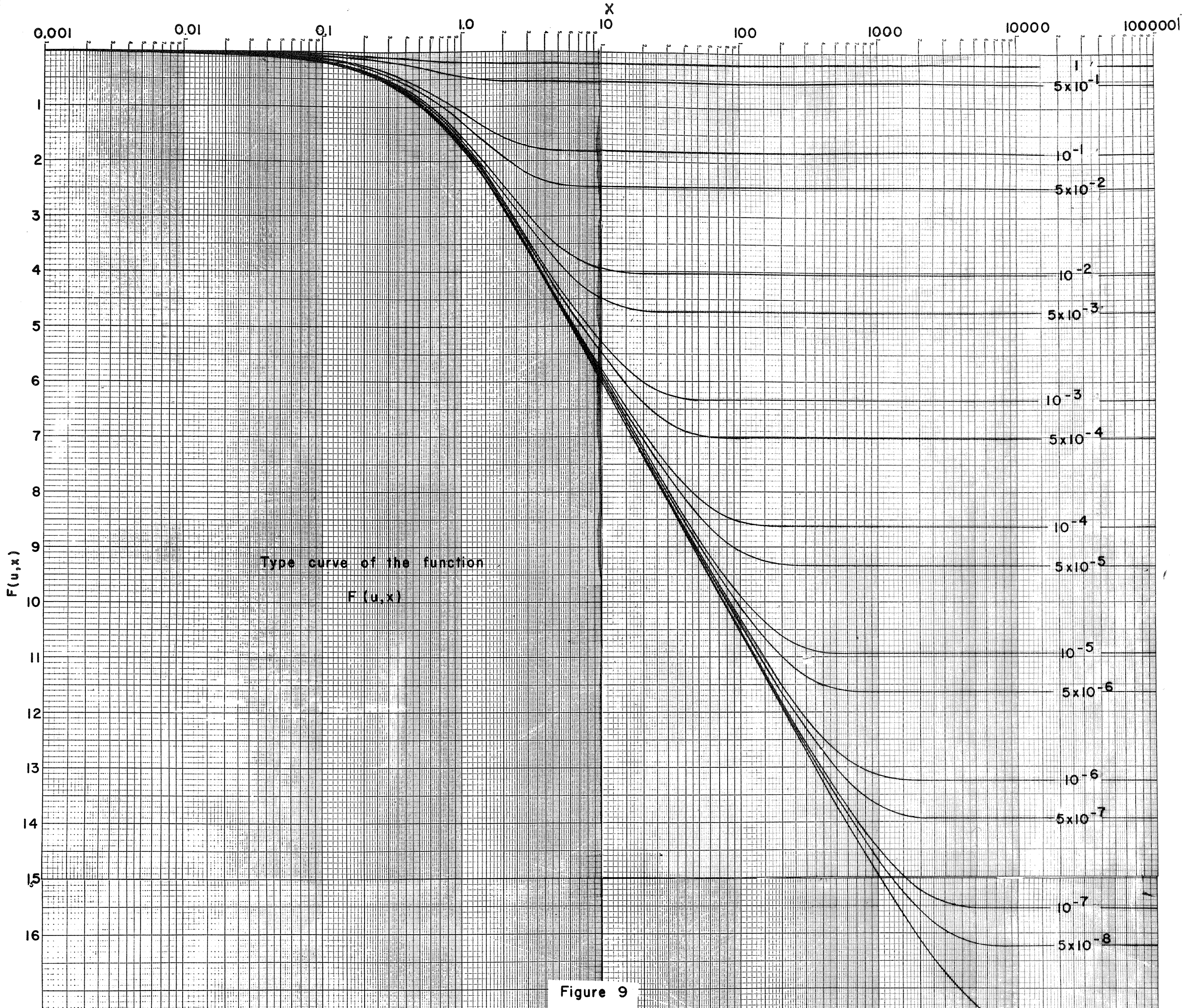

X
0.001
0.01
0.1
1.0
10
100
1000
10000
100000
F(u,x)
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
Type curve of the function
F (u,x)
1
5x10^-1
10^-1
5x10^-2
10^-2
5x10^-3
10^-3
5x10^-4
10^-4
5x10^-5
10^-5
5x10^-6
10^-6
5x10^-7
10^-7
5x10^-8
10^-8

Figure 9

APPLICATIONS

In the quantitative study of ground-water resources, it is essential to determine the field values of the so-called formation constants; namely, the hydraulic conductivity and the specific storage. Based on the solutions obtained in each of the flow systems discussed above in the section on theory, methods are outlined below for determining the formation coefficients by using data from an aquifer test. The methods can be classified as: (1) the type-curve method, (2) the inflection-point method, and (3) the two-point method.

Type-Curve Method

This method is essentially that of Theis (Wenzel, 1942). The appropriate type curve is a plot of the related function of the flow system versus $\frac{1}{u}$ on log-log paper. The observational data are plotted against t on log-log paper of the same scale. The two curves are matched, the axis of the two sheets being kept parallel. A matching point anywhere on the two sheets is selected, and the usual procedure is followed in computing for the formation coefficients.

The systems of flow and their appropriate functions for the type curves are listed below:

Flow system	Observational well	Function	Coefficient obtained	Equation
Wells just tapping the aquifer				
Case 1: Wells of constant discharge	Just tapping the aquifer	$\operatorname{erfc}(\sqrt{u})$	K & S_s	(19)
	Perforated from top to bottom	$F(u, \frac{b'}{r})$	K & S_s	(20)
Case 2: Wells of constant drawdown	Just tapping the aquifer	$\operatorname{erfc}(\sqrt{u}_{r-r_w})$	r_w, *	(28)
Wells of finite depth	Just tapping the aquifer	$F(u, \frac{b}{r})$	K & S_s	(41), $(z = o)$
	Perforated from top to bottom	$E(u, \frac{b}{r}, \frac{b'}{r})$	K & S_s	(42)

* The coefficients K and S_s can be determined by using equation (30). A plot of Q against $\frac{1}{\sqrt{t}}$ is a straight line whose intercept with the Q axis $= 2\pi K R_w S_w$, from which K is obtained. The slope of the straight line

$$= 2\pi K R_w S_w \cdot \frac{R_w}{\sqrt{\pi}} \sqrt{\frac{S_s}{K}},$$ from which S_s is obtained.

It should be observed that the type curve is not the same for all the observation wells used in an aquifer test; rather, each observation well has its own type curve. This necessitates a lengthy procedure; however, it is the only method yet devised. In actual application, the early part and probably the latter part of the observed data may have to be rejected. During the early period of pumping, the probable variation of the formation coefficients will cause the observed data to deviate from the type curve. The deviation of the data collected during the latter period of pumping may be due to the fact that the flow system is beginning to be influenced by the impermeable bed of the aquifer.

Inflection-Point Method

This method can be used only if a steady-state flow is essentially attained and if both the pumped and the observed wells are just tapping the aquifer.

The method is based on the observation that the curve of the time-drawdown semilog plot has an inflection point at $u_i = 1/2$. This being the case, the drawdown at the inflection point s_i is then equal to $0.317\, s_m$, where s_m is the maximum drawdown. Therefore, if the maximum drawdown can be extrapolated from the time-drawdown semilog plot, the point of inflection can be located on the graph by scaling the value $s_i = 0.317\ s_m$. The value of K is obtained by using equation (21), and that of S_s by using the relation $u_i = 1/2$.

If the flow system is that of a flowing well or a spring that just taps the aquifer, the same procedure can be followed, subject to the same conditions, to obtain the effective radius R_w and the formation coefficients, by using equations (32), (34), and (28) successively.

Two-Point Method

This method can be used if the observation well is open only at the top of the aquifer. It can also be used to analyze data from the pumping well, provided that the well losses can be estimated or are very small. The method is based on the following analysis:

Let t_1 be any convenient time, and let n be any number (in practice n is conveniently taken to be 2). Then, if $s_1(t_1)$ and $s_2(nt_1)$ are the values of the drawdown at time t_1 and nt_1, it follows from equation (41) that

$$\frac{s_1(t_1)}{s_2(nt_1)} = \frac{F_1(u,\ b/r)}{F_2(u/n,\ b/r)} = f(u,\ \frac{b}{r}).$$

Here $\frac{s_1}{s_2}$ and $\frac{b}{r}$ are known, and so u can be read from a table or a graph of $f(u, \frac{b}{r})$, which function is tabulated in Table 2 for $n = 2$. When u is found, $F_1(u, \frac{b}{r})$ is obtained from Table 1. Hence K can be computed from

$$K = \frac{Q}{4\pi b s_1} F_1(u, \frac{b}{r}), \quad \text{and } S_s \text{ from} \quad u = \frac{r^2 S_s}{4Kt_1}.$$

The procedure is carried out for several pairs of points (usually t_1, $2t_1$; $2t_1$, $4t_1$; $4t_1$, $8t_1$; . . .); the computed values of K and S_s then are averaged.

Table-2 f(u,x)

$$f(u,x) = \frac{F(u,x)}{F(u/2,x)}$$

x \ u	1×10^{-7}	1×10^{-6}	1×10^{-5}	1×10^{-4}	1×10^{-3}	1×10^{-2}	1×10^{-1}	1
1	1	.999	.999	.996	.988	.959	.851	.437
2	1	.999	.998	.995	.985	.945	.815	.397
3	1	.999	.998	.994	.983	.939	.785	.391
4	1	.999	.998	.994	.979	.929	.764	
5	1	.999	.998	.993	.976	.919	.750	
6	1	.999	.997	.992	.973	.911	.744	
7	1	.999	.997	.991	.971	.902	.741	
8	1	.999	.997	.990	.968	.895	.740	
9	1	.999	.997	.989	.965	.888	.739	
10	1	.999	.996	.988	.963	.883		
15	1	.998	.995	.985	.951	.864		
20	.999	.998	.994	.981	.940	.857		
25	.999	.998	.993	.978	.931	.855		
30	.999	.997	.992	.975	.924	.854		
35	.999	.997	.991	.972	.918			
40	.999	.997	.990	.969	.913			
45	.999	.997	.989	.969	.910			
50	.999	.996	.988	.963	.907			
55	.999	.996	.987	.960	.906			
60	.999	.996	.986	.958	.904			
65	.999	.996	.985	.955	.903			
70	.998	.995	.985	.953	.902			
75	.998	.995	.984	.951	.902			
80	.998	.995	.983	.949	.902			
85	.998	.995	.982	.947	.901			
90	.998	.994	.981	.945				
95	.998	.994	.980	.943				
100	.998	.993	.980	.942				
150	.997	.991	.972	.931				
200	.996	.989	.965	.927				
250	.996	.986	.960	.926				
300	.995	.984	.955					
350	.994	.982	.951					
400	.994	.980	.948					
450	.993	.978	.946					
500	.992	.976	.944					
550	.992	.974	.943					
600	.991	.972	.942					
650	.990	.970	.941					
700	.990	.968	.941					
750	.989	.967	.941					
800	.988	.966	.941					
850	.988	.965	.940					
900	.987	.964						
950	.987	.962						
1000	.986	.961						
1500	.980	.954						
2000	.976	.951						
2500	.971	.950						
3000	.968							
3500	.965							
4000	.963							
4500	.961							
5000	.960							
5500	.959							
6000	.958							
6500	.958							
7000	.958							
7500	.957	.950	.940	.926	.901	.854	.739	.391

EXAMPLES

Aquifer test data obtained are used to illustrate application of the type-curve method. Two pumped wells have been treated. The effective radius is taken as the radius of the gravel packing, and the well losses are neglected. It should be observed that because of these assumptions, the results of the analysis undoubtedly are in error. The analysis is carried out for the purpose of illustrating the method only.

The first well is located at SE1/4SW1/4SE1/4 sec. 20, T. 11 N., R. 30 E., Quay County, its casing is 12 in. diameter, with 6 in. gravel pack, and has a perforated depth of 30 ft. The well was pumped at an average rate of 12 gallons per minute. Figure 10 is a plot of the type curve using equation (38) with $z = o$. The type curve is matched on the observed drawdown curve. A matching point was chosen having the coordinates; $F = 1.2$; $\frac{1}{u} = 25.6$; $s = 5$ ft; $t = 20$ min. The permeability and specific storage can be found as

$$K = \frac{114.6 \times 12 \times 1.2}{30 \times 5} = 11.0 \text{ gal / day / ft}^2$$

$$S_s = \frac{11 \times 20}{1440 \times 25.6 \times 1.87} = .0032 \text{ ft}^{-1}$$

The second well is located at SE1/4NE1/4SW1/4 sec. 12, T. 24 S., R. 11 W., Luna County, its casing is 12 in. diameter, and has a perforated depth of 100 ft. The well was pumped at an average rate of 374 gal / min. Figure 11 is a plot of the type curve using equation (38) with z = o. The type curve is matched on the observed drawdown curve. A matching point was chosen having the coordinates; $F = 4.8$; $\frac{1}{u} = 2.8 \times 10^4$; $s = 10$ ft; $t = 100$ min. The permeability and specific storage can be found as

$$K = \frac{114.6 \times 374 \times 4.8}{10 \times 100} = 205.7 \text{ gal / day / ft}^2$$

$$S_s = \frac{205.7 \times 100}{1440 \times 2.8 \times 10^4 \times 1.87} = .00028 \text{ ft}^{-1}$$

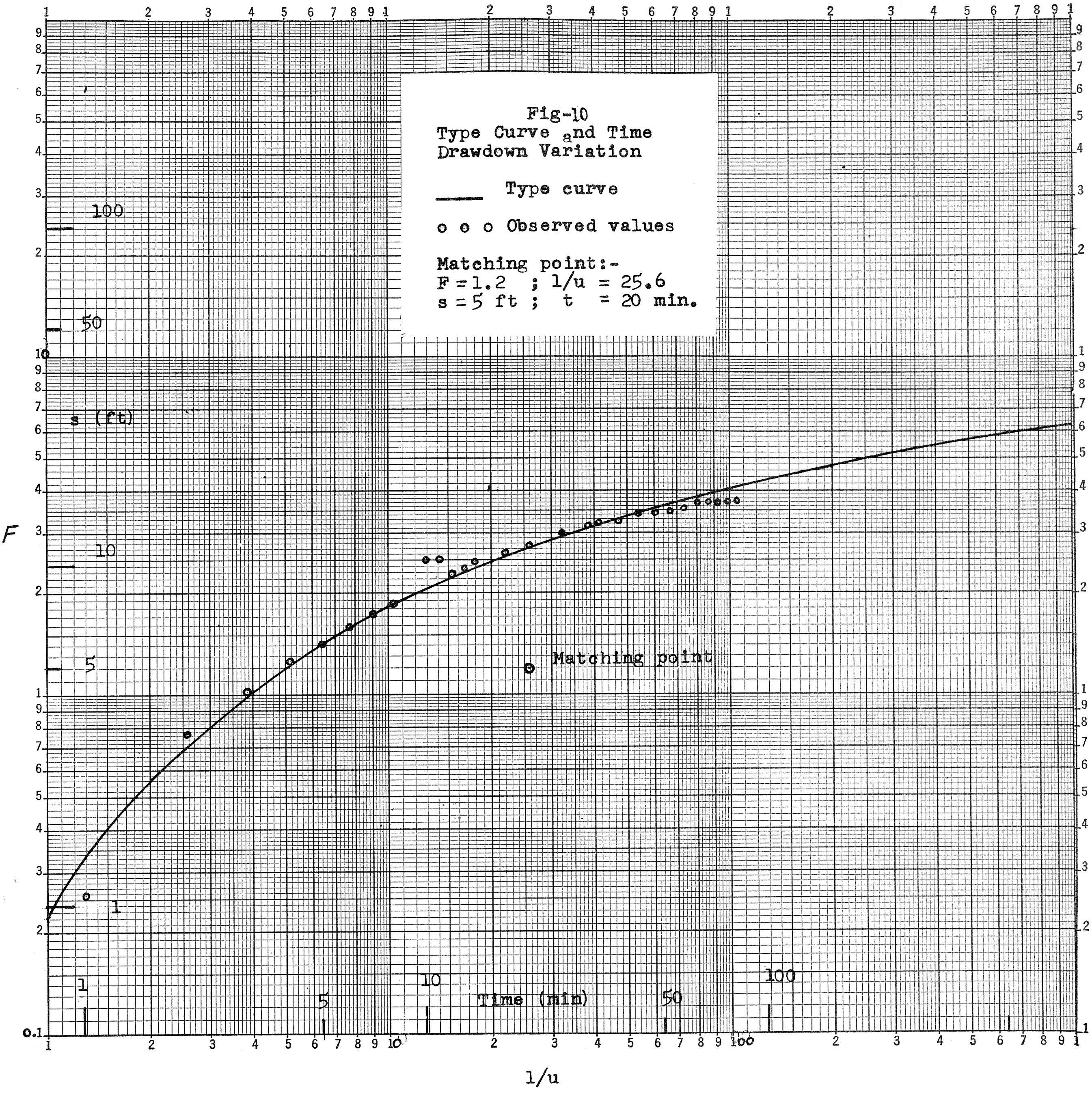


Fig-10
Type Curve and Time
Drawdown Variation
Type curve
o o o Observed values
Matching point:-
F = 1.2 ; 1/u = 25.6
s = 5 ft ; t = 20 min.
Matching point
s (ft)
F
Time (min)
1/u

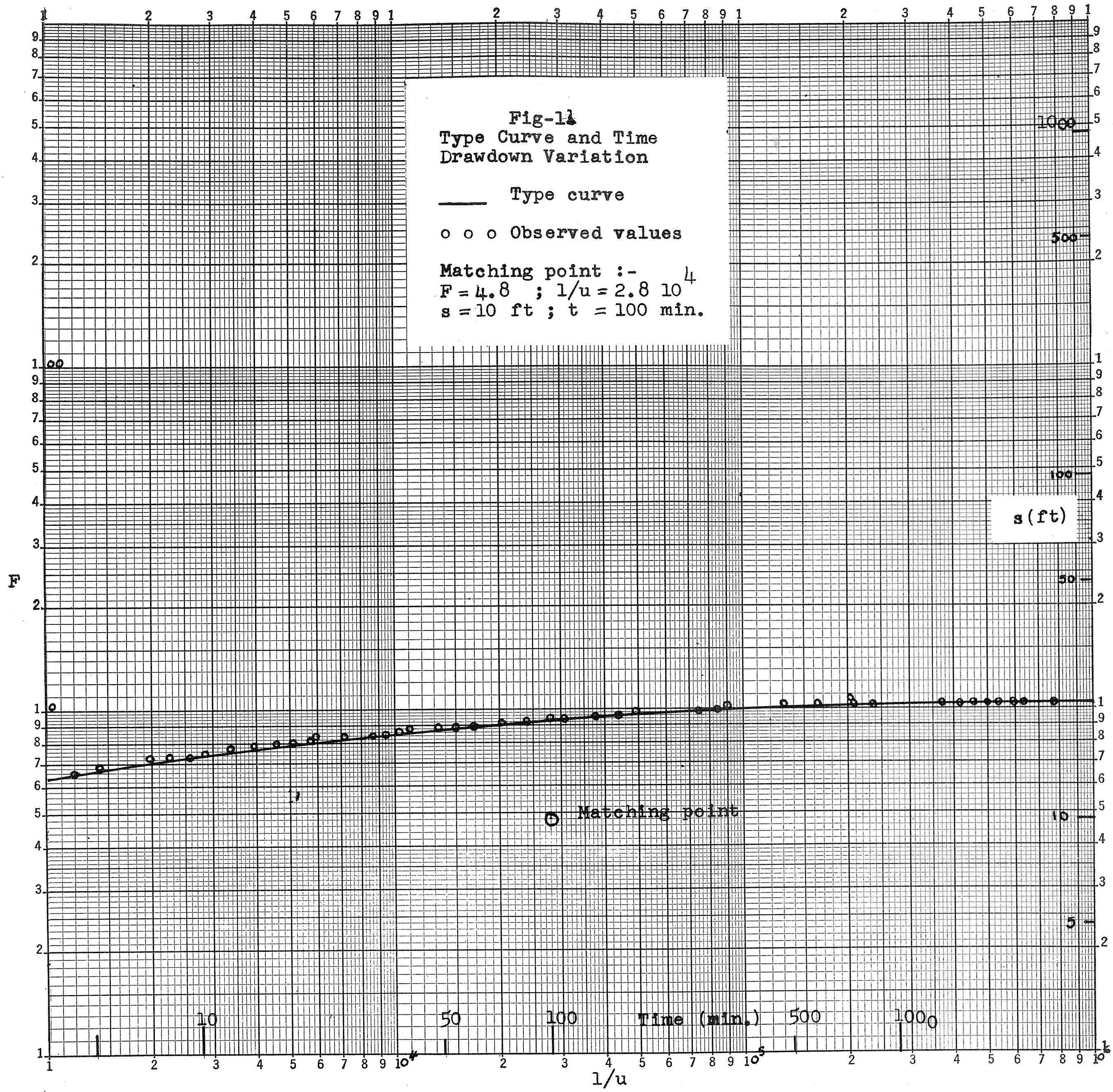


Fig-11
Type Curve and Time
Drawdown Variation
Type curve
o o o Observed values
Matching point :-
F = 4.8 ; 1/u = 2.8 10^4
s = 10 ft ; t = 100 min.
Matching point
s(ft)
F
1/u
Time (min.)
10
50
100
500
1000
5
10
50
100
500
1000

REFERENCES

BABBIT, H. E., and D. H. COLDWELL, The free surface around and interface between gravity wells, Univ. of Illinois Bull. 374, Jan. 1948.

BATEMAN, H. (Erdelyi, A., editor), Bateman manuscript project, McGraw-Hill Book Co., New York, 1954.

BOULTON, N. S., The drawdown distribution of the water table under nonsteady conditions near a pumped well in an unconfined formation, Institution Civil Eng., London, August 1954.

BUTLER, S. S., Engineering Hydrology, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1957.

CHURCHILL, R. V., Operational mathematics, McGraw-Hill Book Co., New York, 1958.

DWIGHT, H. B., Mathematical tables, Dover Publications, Inc., New York, 1958.

HANTUSH, M. S., Nonsteady flow to a well partially penetrating an infinite leaky aquifer, Proc. Iraqi Sci. Soc., 1, 10-19, 1957.

KOZNEY, J., Wasserkraft u. Wasserwirtschaft, 28, 88-101, 1933.

MUSKAT, M., Flow of homogeneous fluids through porous media, McGraw-Hill Book Co., New York, 1937.

PORCHET, M., Hydrodynamics des puits, Annales du ministere de l'agriculture, 60, 203, 1930.

THEIS, C. V., The relation between the lowering of the piezometric surface and the rate and duration of discharge of a well using ground-water storage, Trans. Amer. Geophys. Union, 16, 519-524, 1935.

WATSON, G. N., Theory of Bessel functions, Macmillan Co., New York, 1944.

WENZEL, L. K., Methods for determining permeability of water-bearing materials, U. S. Geol. Survey, Water-Supply Paper 887, 1942.

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May 13, 1960

TABLE 3 Values of the function F (u, x)*

u \ x	1	2	3	4	5	6	7	8	9	10
5										
6										
7										
8										
9										
1×10^{-5}										
2										
3										
4										
5										
6										
7										
8										
9										
1×10^{-4}										
2										
3										
4										
5									5.335	5.495
6									5.29	5.44
7									5.26	5.40
8									5.22	5.36
9									5.18	5.32
1×10^{-3}			3.423	3.905	4.269	4.558	4.792	4.986	5.151	5.291
2			3.33	3.78	4.11	4.37	4.60	4.78	4.90	5.00
3			3.25	3.69	3.99	4.24	4.46	4.58	4.71	4.80
4			3.21	3.62	3.90	4.13	4.32	4.44	4.56	4.62
5			3.161	3.558	3.839	4.046	4.203	4.321	4.414	4.484
6			3.12	3.49	3.76	3.96	4.10	4.21	4.31	4.35
7			3.07	3.44	3.70	3.88	4.01	4.11	4.21	4.22
8			3.04	3.38	3.64	3.81	3.93	4.02	4.11	4.12
9			3.00	3.34	3.58	3.74	3.86	3.94	4.02	4.03
1×10^{-2}	1.538	2.438	2.969	3.305	3.530	3.685	3.793	3.868	3.921	3.958
2	1.43	2.25	2.72	3.01	3.13	3.19	3.28	3.30		
3	1.36	2.12	2.52	2.79	2.84	2.88	2.91	2.93		
4	1.31	2.02	2.36	2.59	2.62	2.63	2.67	2.69		
5	1.269	1.914	2.216	2.356	2.421	2.449	2.461	2.465		
6	1.22	1.84								
7	1.18	1.76								
8	1.15	1.69								
9	1.11	1.62								
1×10^{-1}	1.080	1.560								
2	.83	1.08								
3	.66	.83								
4	.55	.67								
5	.458	.551								

u \ x	15	20	25	30	35	40
5						
6						
7						
8						
9						
1×10^{-5}						
2						
3						
4						
5				7.713	7.943	8.132
6				7.66	7.90	8.07
7				7.62	7.84	8.02
8				7.58	7.80	7.96
9				7.55	7.76	7.92
1×10^{-4}	6.467	6.928	7.265	7.519	7.718	7.877
2	6.29	6.74	7.05	7.26	7.42	7.52
3	6.17	6.61	6.89	7.06	7.20	7.27
4	6.08	6.49	6.74	6.90	7.00	7.07
5	6.057	6.392	6.606	6.746	6.840	6.903
6	5.95	6.30	6.50	6.62	6.69	6.75
7	5.89	6.22	6.40	6.50	6.57	6.62
8	5.84	6.14	6.31	6.40	6.46	6.51
9	5.80	6.07	6.23	6.31	6.36	6.40
1×10^{-3}	5.760	6.011	6.153	6.233	6.278	6.304
2	5.38	5.50	5.61			
3	5.08	5.16	5.25			
4	4.86	4.91	4.98			
5	4.662	4.712	4.724			
6						
7						
8						
9						
1×10^{-2}						
2						
3						
4						
5						
6						
7						
8						
9						
1×10^{-1}						
2						
3						
4						
5						

u \ x	45	50	55	60	65	70	75	80	85	90	95	100
5										9.934	10.017	10.095
6										9.89	9.96	10.05
7										9.84	9.92	10.00
8										9.81	9.89	9.96
9										9.77	9.85	9.92
1×10^{-5}	8.679	8.855	9.010	9.149	9.274	9.387	9.490	9.584	9.671	9.739	9.823	9.891
2	8.54	8.70	8.84	8.98	9.10	9.18	9.28	9.34	9.42	9.49	9.55	9.60
3	8.44	8.58	8.72	8.84	8.96	9.02	9.12	9.16	9.25	9.30	9.36	9.40
4	8.36	8.50	8.62	8.73	8.83	8.90	8.99	9.02	9.10	9.16	9.19	9.24
5	8.290	8.424	8.538	8.636	8.722	8.796	8.861	8.917	8.967	9.010	9.049	9.082
6	8.23	8.36	8.46	8.55	8.62	8.70	8.76	8.81	8.86	8.89	8.93	8.96
7	8.17	8.30	8.39	8.46	8.54	8.60	8.67	8.71	8.76	8.79	8.82	8.85
8	8.12	8.24	8.33	8.40	8.47	8.52	8.58	8.62	8.67	8.68	8.72	8.74
9	8.06	8.17	8.26	8.33	8.40	8.44	8.50	8.54	8.58	8.60	8.67	8.68
1×10^{-4}	8.006	8.112	8.200	8.272	8.333	8.383	8.425	8.460	8.491	8.515	8.535	8.552
2	7.62	7.68	7.74	7.82	7.86	7.87	7.92	7.92	7.92			
3	7.34	7.39	7.43	7.50	7.53	7.53	7.54	7.56	7.56			
4	7.13	7.16	7.20	7.26	7.26	7.27	7.27	7.28	7.28			
5	6.945	6.973	6.985	7.004	7.012	7.017	7.020	7.022	7.024			
6												
7												
8												
9												
1×10^{-3}												
2												
3												
4												
5												
6												
7												
8												
9												
1×10^{-2}												
2												
3												
4												
5												
6												
7												
8												
9												
1×10^{-1}												
2												
3												
4												
5												

* The values of the function for the smaller intervals (not included in Table 1) are obtained by graphical interpolation.

TABLE 3 Values of the function F (u, x)* (continued)

u \ x	150	200	250	300	350	400	450	500	550	600
1×10^{-8}										
2										
3										
4										
5										
6										
7										
8										
9										
1×10^{-7}							13.284	13.460	13.615	13.754
2							13.15	13.30	13.45	13.58
3							13.05	13.20	13.34	13.44
4							12.97	13.10	13.24	13.34
5				12.317	12.548	12.737	12.895	13.029	13.143	13.241
6				12.27	12.50	12.68	12.83	12.96	13.06	13.15
7				12.23	12.45	12.62	12.77	12.89	12.99	13.08
8				12.19	12.40	12.57	12.71	12.83	12.93	13.00
9				12.16	12.36	12.52	12.65	12.77	12.86	12.93
1×10^{-6}	11.070	11.534	11.870	12.123	12.323	12.482	12.611	12.717	12.805	12.877
2	10.93	11.34	11.64	11.85	12.01	12.13	12.24	12.30	12.37	12.41
3	10.82	11.20	11.49	11.65	11.80	11.88	11.96	12.00	12.07	12.08
4	10.74	11.09	11.34	11.50	11.60	11.68	11.74	11.76	11.80	11.82
5	10.660	10.996	11.210	11.350	11.444	11.508	11.550	11.578	11.597	11.609
6	10.60	10.91	11.10	11.23	11.31	11.37				
7	10.53	10.82	11.00	11.12	11.19	11.23				
8	10.47	10.74	10.92	11.01	11.08	11.12				
9	10.42	10.67	10.84	10.92	10.97	11.01				
1×10^{-5}	10.363	10.615	10.757	10.837	10.883	10.908				
2	9.97	10.11	10.22							
3	9.68	9.78	9.81							
4	9.45	9.52	9.53							
5	9.262	9.312	9.324							

u \ x	650	700	750	800	850	900	950	1000	1500	2000	2500	3000	3500
1×10^{-8}									15.675	16.139	16.475	16.729	16.928
2									15.53	15.95	16.23	16.46	16.62
3									15.43	15.81	16.07	16.25	16.40
4									15.33	15.70	15.94	16.09	16.22
5	14.018	14.137	14.250	14.353	14.450	14.539	14.622	14.700	15.265	15.601	15.815	15.955	16.049
6	13.98	14.10	14.21	14.32	14.41	14.49	14.58	14.65	15.19	15.51	15.71	15.83	
7	13.95	14.07	14.17	14.28	14.37	14.45	14.54	14.61	15.13	15.44	15.62	15.72	
8	13.93	14.04	14.15	14.25	14.34	14.41	14.50	14.57	15.07	15.36	15.53	15.62	
9	13.90	14.02	14.12	14.22	14.30	14.38	14.46	14.53	15.02	15.29	15.44	15.53	
1×10^{-7}	13.879	13.992	14.094	14.189	14.276	14.344	14.428	14.496	14.968	15.220	15.362	15.442	
2	13.69	13.80	13.88	13.96	14.03	14.09	14.18	14.20					
3	13.55	13.65	13.72	13.79	13.86	13.91	13.98	14.00					
4	13.43	13.52	13.58	13.65	13.70	13.75	13.82	13.83					
5	13.327	13.401	13.465	13.522	13.572	13.615	13.654	13.687					
6	13.24	13.30	13.42	13.41	13.45	13.50	13.53	13.57					
7	13.15	13.21	13.32	13.31	13.34	13.40	13.43	13.44					
8	13.07	13.12	13.23	12.22	13.26	13.30	13.33	13.34					
9	13.00	13.05	13.14	13.14	13.16	13.21	13.23	13.25					
1×10^{-6}	12.928	12.979	13.030	13.065	13.095	13.120	13.140	13.157					
2													
3													
4													
5													
6													
7													
8													
9													
1×10^{-5}													
2													
3													
4													
5													

* The values of the function for the smaller intervals (not included in Table 1) are obtained by graphical interpolation.