Ph.D. Preliminary Examination in Numerical Analysis
Department of Mathematics
New Mexico Institute of Mining and Technology
August 12, 2019, 9 AM – 1 PM, Weir 202

1. This exam is four hours long.
2. You need a scientific calculator for this exam.
3. Work out all six problems.
4. Start the solution of each problem on a new page.
5. Number all of your pages.
6. Sign your name on the following line and put the total number of pages.
7. Use this sheet as a coversheet for your papers.

NAME: ________________________________ **No. of pages:**__________

Problem 1.

Consider the function

$$F(x) = \tan^{-1}(x)$$

on an interval $a \le x \le b$, where $0 < a < b$. Show that function $F(x)$ is contractive on this interval and find the smallest value of λ such that

$$|F(x) - F(y)| \le \lambda |x - y|$$

for every x and y in the interval $[a, b]$.

Problem 2.

a) Prove that unitarily similar matices have the same eigenvalues.

b) Let A and B be unitarily similar matrices. Express eigenvectors of B in terms of eigenvectors of A.

Problem 3.

Let A be $m \times n$ real matrix, $m \ge n$, and let A have rank n. Let $b \in R^m$. Prove the following statements:

a) There exists unique $x \in R^n$ minimizing $\|Ax - b\|_2^2$, where $\|\cdot\|_2$ is the 2-norm.

b) The matrix $A^T A$ is invertible and $x = (A^T A)^{-1} A^T b$.

Problem 4.

Find the constants c_0, c_1 and x_1 so that the quadrature formula

$$\int_0^1 f(x)dx = c_0 f(0) + c_1 f(x_1)$$

has the highest possible degree of precision. What is the degree of precision of the quadrature formula?

Problem 5.

a) State the theorem on convergence of the fixed point iteration $x_{n+1} = g(x_n)$ with a function $g(x)$ defined on the interval $[a, b]$.

b) Let $\{x_n\}$ be the convergent sequence generated by the fixed point iteration, let x_* be a fixed point of $g(x)$, and let $k = \max_{x \in [a,b]} |g(x)|$. Give an estimate of the error $|x_n - x_*|$ in terms of k.

c) Consider the function $g(x) = x^2 + \frac{3}{16}$. Find all fixed points of $g(x)$.

d) For each of the fixed points, explain why or why not the fixed point iteration with $g(x)$ is guaranteed to converge. If fixed point iteration is guaranteed to converge, specify an interval $[a, b]$ such that, for any initial guess $x_0 \in [a, b]$, the fixed point iteration produces a convergent sequence.

e) For each convergent fixed point iteration, determine how many iterations are required to reduce the error by a factor of 10.

Problem 6.

a) Describe the power method for computing the dominant eigenvalue and a corresponding eigenvector of a matrix.

b) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}.$$

Is the matrix semi-simple (diagonalizable)? Does it have a dominant eigenvalue?

c) Demonstrate convergence of the power method by performing four iterations on matrix A with the initial guess $v = (1, 0)^T$.

d) Compute the relative errors of the obtained approximations using the ∞-norm.

Solutions

Problem 1.

Consider the function

$$F(x) = \tan^{-1}(x)$$

on an interval $a \leq x \leq b$, where $0 < a < b$. Show that function $F(x)$ is contractive on this interval and find the smallest value of λ such that

$$|F(x) - F(y)| \leq \lambda |x - y|$$

for every x and y in the interval $[a, b]$.

Solution.

Take any two points x and y in $[a, b]$. Recall that the derivative of $\tan^{-1}(x)$ is $1/(1+x^2)$. By the mean value theorem,

$$F(x) - F(y) = F'(\xi)(x - y)$$

for some ξ in the interval between x and y. Thus

$$|F(x) - F(y)| = (1/(1 + \xi^2))|x - y|$$

Since the largest possible value of $1/(1 + \xi^2)$ occurs at $\xi = a$,

$$|F(x) - F(y)| \leq (1/(1 + a^2))|x - y|$$

for every x and y in $[a, b]$. Note that $1/(1 + a^2) < 1$. Thus F is a contractive mapping on $[a, b]$ and the best possible value of λ is $1/(1 + a^2)$.

Problem 2.

a) Prove that unitarily similar matices have the same eigenvalues.

b) Let A and B be unitarily similar matrices. Express eigenvectors of B in terms of eigenvectors of A.

Solution.

Let λ be any eigenvalue of A. Then

$$\det(A - \lambda I) = 0.$$

$$\det(Q)\det(A - \lambda I)\det(Q^*) = 0.$$
$$\det(Q(A - \lambda I)Q^*) = 0.$$
$$\det(QAQ^* - \lambda QQ^*) = 0.$$

But $QQ^* = I$, so

$$\det(QAQ^* - \lambda I) = 0.$$

Since $B = QAQ^*$,

$$\det(B - \lambda I) = 0.$$

Thus λ is an eigenvalue of B.

Conversely, if λ is an eigenvalue of B, then since

$$B = QAQ^*$$

we have

$$Q^*BQ = A.$$

By the same arguement as above, if λ is an eigenvalue of B, then λ is an eigenvalue of A.

If v is an eigenvector of A with eigenvalue λ, then

$$Av = \lambda v$$

$$Q^*BQv = \lambda v$$
$$B(Qv) = \lambda(Qv)$$

Thus Qv is an eigenvector of B with eigenvalue λ.

Problem 3.

Let A be $m \times n$ real matrix, $m \geq n$, and let A have rank n. Let $b \in R^m$. Prove the following statements:

a) There exists unique $x \in R^n$ minimizing $\|Ax - b\|_2^2$, where $\|\cdot\|_2$ is the 2-norm.

b) The matrix $A^T A$ is invertible and $x = (A^T A)^{-1} A^T b$.

Solution.

a) Let the SVD decomposition of A be $A = U\Sigma V^T$, where $U \in R^{m\times m}$ and $V \in R^{n\times n}$ are orthogonal matrices, $\Sigma \in R^{m\times n}$ has non-zero entries $Sigma_{i,i} = \sigma_i$ (for $i \leq n$, where σ_i are the singular values).

$$\|Ax - b\|_2^2 = \|U^T(Ax - b)\|_2^2 = \|U^T Ax - U^T b\|_2^2 = \|\Sigma(V^T x) - U^T b\|_2^2.$$

Let $y = V^T x \in R^n$ and $c = U^T b \in R^m$, then

$$\|Ax - b\|_2^2 = \|\Sigma y - c\|_2^2 = \sum_{i=1}^{n} |\sigma_i y_i - c_i|^2 + \sum_{i=n+1}^{m} |c_i|^2.$$

Therefore, $\|Ax - b\|_2^2$ is minimized if and only if $\sigma_i y_i - c_i = 0$ (i.e., $y_i = c_i/\sigma_i$) for $i = 1, \ldots, n$. Consequently, the solution $x = Vy$ is uniquely determined.

b) First show the invertibility.

$$A^T A = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T \Sigma V^T. \tag{1}$$

The matrix $\Sigma^T\Sigma \in R^{n\times n}$ is a diagonal matrix with all n non-zero main-diagonal entries σ_i^2 ($i = 1, \ldots, n$), so it is invertible. Therefore, $(A^T A)^{-1} = V(\Sigma^T\Sigma)^{-1}V$. The least squares problem $\|Ax - b\|_2^2$ is minimized if and only if $Ax - b \in N(A^T)$ (i.e., $A^T(Ax - b) = 0$). Then the solution x satisfies $A^T Ax = A^T b$. Since $A^T A$ is invertible, x can be solved as $x = (A^T A)^{-1} A^T b$.

Problem 4.

Find the constants c_0, c_1 and x_1 so that the quadrature formula

$$\int_0^1 f(x)dx = c_0 f(0) + c_1 f(x_1)$$

has the highest possible degree of precision. What is the degree of precision of the quadrature formula?

Solution.

Let $f(x) = 1$, then $\int_0^1 1dx = c_0 + c_1$.
Let $f(x) = x$, then $\int_0^1 xdx = c_0 0 + c_1 x_1$.
Let $f(x) = x^2$, then $\int_0^1 x^2 dx = c_0 0 + c_1 x_1^2$.
Simply the equations, we end up with

$$\begin{aligned} 1 &= c_0 + c_1 \\ \frac{1}{2} &= c_1 x_1. \\ \frac{1}{3} &= c_1 x_1^2. \end{aligned}$$

The values of the constants are

$$c_0 = \frac{1}{4}, c_1 = \frac{3}{4}, x_1 = \frac{2}{3}.$$

Problem 5.

a) State the theorem on convergence of the fixed point iteration $x_{n+1} = g(x_n)$ with a function $g(x)$ defined on the interval $[a, b]$.

b) Let $\{x_n\}$ be the convergent sequence generated by the fixed point iteration, let x_* be a fixed point of $g(x)$, and let $k = \max_{x\in[a,b]} |g(x)|$. Give an estimate of the error $|x_n - x_*|$ in terms of k.

c) Consider the function $g(x) = x^2 + \frac{3}{16}$. Find all fixed points of $g(x)$.

d) For each of the fixed points, explain why or why not the fixed point iteration with $g(x)$ is guaranteed to converge. If fixed point iteration is guaranteed to converge, specify an interval $[a, b]$ such that, for any initial guess $x_0 \in [a, b]$, the fixed point iteration produces a convergent sequence.

e) For each convergent fixed point iteration, determine how many iterations are required to reduce the error by a factor of 10.

Solution.

Theorem.

Let $g(x)$ be a continuously differentiable function on the interval $[a, b]$ such that

$$g(x) \in [a, b], \ \forall x \in [a, b],$$

let $k = \max_{x\in[a,b]} |g(x)|$. If $k < 1$, then function $g(x)$ has a unique fixed point $x_* \in [a, b]$, the fixed point iteration $x_{n+1} = g(x_n)$ converges to x_* for any initial guess $x_0 \in [a, b]$, and the following error estimate holds

$$|x_n - x_*| \leq k^n |x_0 - p|.$$

The number of iterations required to reduce the error by a factor of 10 is at most

$$\left[-\frac{1}{\log_{10} k} \right] + 1.$$

Other acceptable error estimates are

$$|x_n - x_*| \leq k^n \max\{x_0 - a, \ b - x_0\},$$

and

$$|x_n - x_*| \leq \frac{k^n}{1-k} |x_1 - x_0|.$$

Function $g(x) = x^2 + \frac{3}{16}$ has two fixed points

$$x_1 = \frac{1}{4} \text{ and } x_2 = \frac{3}{4}.$$

Since $g'(x_2) = 2x_2 = \frac{3}{2} > 1$, the assumption $|g'(x)| \leq k < 1$ of the fixed point theorem is not satisfied, and the theorem can not be applied for the fixed point x_2. Therefore, fixed point iteration is not guaranteed to converge to x_2.

We now show that function $g(x)$ satisfies the assumptions of the fixed point theorem for a fixed point $x_* = x_1 = \frac{1}{4}$ on the interval $[0, \frac{3}{8}]$. Since $g(0) = \frac{3}{16} < \frac{3}{8}$, $g(\frac{3}{8}) = \frac{21}{64} < \frac{3}{8}$, and $g'(x) \geq 0$ for $x \in [0, \frac{3}{8}]$, we have $g([0, \frac{3}{8}]) \in [0, \frac{3}{8}]$. Function $g(x)$ is continuously differentiable on the interval $[0, \frac{3}{8}]$, and

$$\max_{[0,\frac{3}{8}]} |g'(x)| = \max_{[0,\frac{3}{8}]}(2x) = (2)\frac{3}{8} = \frac{3}{4} < 1.$$

Applying the fixed point theorem with $k = \frac{3}{4}$, conclude that fixed point iterations converge to the fixed point $x_* = \frac{1}{4}$ for any initial guess $x_0 \in [0, \frac{3}{8}]$. The number of itertions to reduce the initial error by a factor of 10 is at most

$$\left\lfloor -\frac{1}{\log_{10} k} \right\rfloor + 1 = \left\lfloor -\frac{1}{\log_{10} \frac{3}{4}} \right\rfloor + 1 = 8 + 1 = 9.$$

Problem 6.

a) Describe the power method for computing the dominant eigenvalue and a corresponding eigenvector of a matrix.

b) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}.$$

Is the matrix semi-simple (diagonalizable)? Does it have a dominant eigenvalue?

c) Demonstrate convergence of the power method by performing four iterations on matrix A with the initial guess $v = (1, 0)^T$.

d) Compute the relative errors of the obtained approximations using the ∞-norm.

Solution.

a) The power method iterations are formulated as follows:

$$v_{k+1} = \frac{1}{\sigma_{k+1}} A v_k,$$

where the scaling factor σ_{k+1} is the entry of Av_k with the largest magnitude.

b) Eigenvalues and the corrsponding eigenvectors of matrix A are

$$\lambda_1 = 10, \ \lambda_2 = 2,$$

and

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Since A is 2×2, and it has two distinct eigenvalues, the matrix A has two linearly independent eigenvectors; hence, A is semi-simple. In fact, A is diagonalizable; that is, semi-simple, because it is symmetric: $A^T = A$. Since $\lambda_1 > \lambda_2$, $\lambda_1 = 10$ is the dominant eigenvalue of A, and $v_1 = (1, 1)^T$ is a corresponding dominant eigenvector.

c) Performing iterations with a given matrix A and the initial guess v, obtain

k	σ_k	$(v_k)_2$
0	1	0
1	6	0.666666666666667
2	8.666666666666666	0.923076923076923
3	9.692307692307693	0.984126984126984
4	9.936507936507937	0.996805111821086

In the table, $(v_k)_2$ is the second entry of the k-approximation to a dominant eigenvector $v = (1, 1)^T$.

d) The relative errors are

$$\lambda_1: \ \frac{|10 - 9.936507936507937|}{10} \approx 0.006349$$

and

$$v_1: \ \frac{|1 - 0.996805111821086|}{1} \approx 0.003195$$