(Answer all six problems)

1. Describe the solution of the least squares problem

\[
\min_{\bar{x} \in \mathbb{R}^n} \|A\bar{x} - \bar{b}\|_2
\]

by the QR factorization method where \( A \in \mathbb{R}^{m \times n} \), \( m > n \) and the rank of \( A \) is known.

2. Develop the steepest descent method for solving the linear system,

\[
A\bar{x} = \bar{b}
\]

with \( A \in \mathbb{R}^{n \times n} \), a positive definite matrix. Describe the selection of the search direction, the steplength, and discuss the update of the residual.

3. Let

\[
f(x) = \frac{1 - \sin x}{\pi/2 - x}.
\]

(a) Explain why straight forward evaluation of \( f(x) \) in double precision floating point arithmetic produces wildly inaccurate answers for \( x \) near \( \pi/2 \).

(b) Derive an alternative formula that is much more accurate for \( x \) near \( \pi/2 \).

4. Consider the equation

\[x + \log x = 0.
\]

Here log is the logarithm to the base \( e \). This equation has a solution somewhere near \( x = 0.5 \). Derive a fixed point iteration scheme (other than the Newton’s method) for solving this equation. Show that your fixed point iteration will converge if \( x_0 \) is sufficiently close to the root. Starting with \( x_0 = 0.5 \), use your iteration to solve the equation, obtaining a root accurate to 4 digits.

5. Given the ODE

\[
y'(t) = f(t, y(t))
y(0) = y_0
\]

(a) Derive the Adams-Bashforth Two-Step Explicit Method

\[w_{i+1} = w_i + \frac{h}{2} \left[ 3f(t_i, w_i) - f(t_{i-1}, w_{i-1}) \right]
\]

where \( h > 0 \), and \( t_i = ih, i = 0, 1, \ldots \).
(b) Show that the truncation error is given by

\[ \tau_{i+1} = \frac{5}{12} y'''(\mu_i) h^2 \]

where \( \mu_i \in (t_i, t_{i+1}) \).

6. Let \( f(x) \in C^4[a, b] \). For any \( y, z \in R \), Simpson’s rule is given by

\[ S(y, z) = \frac{h}{3} \left[ f(y) + 4f\left(\frac{y+z}{2}\right) + f(z) \right] \]

where \( h = \frac{z-y}{2} \), and, in particular, satisfies

\[ \int_a^b f(x) \, dx = S(a, b) - \frac{h^5}{90} f^{(4)}(\mu) \]

for some \( \mu \in [a, b] \). The composite Simpson’s rule satisfies

\[ \int_a^b f(x) \, dx = S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right) - \frac{b-a}{180} \left(\frac{h}{2}\right)^4 f^{(4)}(\tilde{\mu}) \]

for some \( \tilde{\mu} \in [a, b] \), and where \( h = \frac{b-a}{2} \).

(a) By equating the above relations and assuming that \( f^{(4)}(\mu) = f^{(4)}(\tilde{\mu}) \) derive an approximatation for the error

\[ E(a, b) = \left| \int_a^b f(x) \, dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right) \right| \]

involving \( S(a, b), S\left(a, \frac{a+b}{2}\right), \) and \( S\left(\frac{a+b}{2}, b\right) \).

(b) Describe how an adaptive composite Simpson’s algorithm can be developed from this error estimate to obtain an approximation to \( \int_a^b f(x) \, dx \) to a desired accuracy of \( \varepsilon > 0 \). You don’t have to be specific about the algorithm, just describe how it would work in general.