Numerical Analysis Qualifying Exam
Mathematics Department
New Mexico Tech
Fall, 2005

(Answer all 6 questions.)

1. Given the definition of the 2–norm of a matrix

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2.$$

Show that

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$$

Hint: Consider the constrained problem

$$\max \|Ax\|_2^2$$

subject to

$$\|x\|_2^2 = 1.$$

Apply the Lagrange multiplier technique to this problem.

2. A finite difference formula for the first derivative is

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2} f''(c)$$

where c is some point in the interval $(x, x+h)$. Suppose that $|f''(z)| \leq M$ on the interval $(x, x+h)$ and that the values of $f(x)$ and $f(x+h)$ can be computed with absolute error less than or equal to ϵ. Derive a bound on the error in $f'(x)$. Find the value of h that optimizes this bound.

3. Cholesky decompositions.

   (a) Formulate the Cholesky decomposition theorem .

   (b) Develop the outer product form of the Cholesky decomposition.

   (c) Use the algorithm in part (b) to test whether the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 8 & 14 \\ 3 & 14 & 50 \end{pmatrix}$$

   is positive definite. If A is positive definite, find its Cholesky factor.

   (d) Estimate the computational complexity of the algorithm for an n by n matrix.

4. Consider the matrix equation, $A\bar{x} = \bar{b}$ and let $A = L + D + U$ where L is strictly lower triangular, D is diagonal, and U is strickly upper triangular.

(a) The Jacobi iteration algorithm creates the sequence of approximations, $\bar{x}^k = \left(x_1^k, x_2^k, \ldots, x_n^k\right)^T$, $k = 0, 1, \ldots$ where $\bar{x}^{k+1} = T\bar{x}^k + \bar{c}$. Derive the Jacobi iteration matrix, T, and vector, $\bar{c}$, in terms of the matrices, L, D, U and vector, $\bar{b}$.

(b) If A is strictly diagonally dominant, show that , $\|T\|_\infty < 1$, and show that $\lim_{k \longrightarrow \infty} \left\|\bar{e}^k\right\|_\infty = 0$ where $\bar{e}^k = \bar{x} - \bar{x}^k$ and thus show that $\lim_{k \longrightarrow \infty} \bar{x}^k = \bar{x}$.

5. Let $f(x) \in C^3(a, b)$, where the interval, (a, b) contains z and $z+3h$, $h > 0$. Given that the forward difference operator, $A(h) = \frac{1}{h}(f(z+h) - f(z))$, satisfies

$$A(h) = f'(z) + \frac{h}{2} f''(z) + \frac{h^2}{6} f'''(z) + O\left(h^3\right)$$

Use Richardson's extraolation method to derive an $O\left(h^3\right)$ formula for $f'(z)$ involving the values, $f(z)$, $f(z+h)$, $f(z+2h)$ and $f(z+3h)$.

6. Obtain the three-step Adams-Bashforth formula

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_{i+1} = w_i + \frac{h}{12}\left[23f_i - 16f_{i-1} + 5f_{i-2}\right], \quad i = 2, 3, 4, \ldots,$$

for the solution of the initial value problem:

$$y' = f(t, y), \quad y(t_0) = \alpha.$$